

# Lecture Notes in Statistics

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K. Krickeberg, I. Olkin, and B. Singer

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Barry C. Arnold  
Enrique Castillo  
José-Mariá Sarabia

Conditionally  
Specified Distributions

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**Authors**

Barry C. Arnold  
Department of Statistics, University of California  
Riverside, CA 92521-0138, USA

Enrique Castillo  
José-Mariá Sarabia  
Departamento de Economía  
Universidad de Cantabria  
Avda. de los Castros, 39005 Santander, Spain

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TO DARREL  
-B.C.A.

TO MARY CARMEN  
-E.C.

TO MY PARENTS  
-J.M.S.

## PREFACE

The concept of conditional specification is not new. It is likely that earlier investigators in this area were deterred by computational difficulties encountered in the analysis of data following conditionally specified models. Readily available computing power has swept away that roadblock. A broad spectrum of new flexible models may now be added to the researcher's tool box. This monograph provides a preliminary guide to these models. Further development of inferential techniques, especially those involving concomitant variables, is clearly called for.

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Barry C. Arnold  
Riverside, California  
USA

Enrique Castillo  
José Mariá Sarabia  
Santander, Cantabria  
Spain

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