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Harmonic Analysis on the Heisenberg Group

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To my family

Scorn not the castle, Architect,
It's nothing but a child's play!
Scorn not the sonnet, Critic,
It's only ignorance on display!

Preface

This monograph deals with various aspects of harmonic analysis on the Heisenberg group. The Heisenberg group is the most well known example from the realm of nilpotent Lie groups and plays an important role in several branches of mathematics, such as representation theory, partial differential equations, several complex variables and number theory. As it is the 'most commutative' among the noncommutative Lie groups, it offers the greatest opportunity for generalising the remarkable results of Euclidean harmonic analysis.

My aim in this work is to demonstrate how standard results of abelian harmonic analysis, such as Plancherel and Paley-Wiener theorems, Wiener-Tauberian theorems, Bochner-Riesz means and multipliers for the Fourier transform, and so on, take shape in the noncommutative setup of the Heisenberg group. Basic results about the representations and the Fourier transform are covered in the first chapter. There are many good texts dealing with these basic results (see Folland [26]) but most of them stop there to develop different topics. Here, however, we pursue a detailed study of the Fourier transform which goes well beyond the basic Stone-von Neumann theorem. We demonstrate the beautiful interplay between the representation theory on the Heisenberg group and the classical expansions in terms of Hermite and Laguerre functions. We prove analogues of Paley-Wiener theorems and Hardy's theorem for the group Fourier transform.

In the second chapter, we develop the spectral theory of the sublaplacian following Strichartz. The eigenfunctions of the sublaplacian are given in terms of the special Hermite functions. There results expansions of functions in terms of these eigenfunctions, sort of a Peter-Weyl theorem for the Heisenberg group. We prove an Abel summability result for these expansions. Then we go on to study the mapping properties of the spectral projections associated to these expansions and prove Müller's restriction theorem. Using this, we study the Bochner-Riesz means associated to the sublaplacian. We also a develop the Littlewood-Paley-Stein theory and prove a weaker version of the multiplier theorem for the sublaplacian.

A study of the group algebra $L^1(H^n/U(n))$ is undertaken in chapter 3 and some applications are given. The Heisenberg group H^n and the unitary group $U(n)$ form a Gelfand pair. We study the elementary spherical functions associated to this pair and prove versions of Wiener-Tauberian theorem. This part of the chapter has some overlap with the work of Faraut-Harzallah [21]. Using the Wiener-Tauberian theorem and the summability result of Strichartz, we study the injectivity of the spherical mean value operator. We also prove a maximal theorem for the spherical means. In the last chapter, we consider the reduced Heisenberg group, and in that context improve some of the theorems treated in previous chapters.

We do not use any major result from the representation theory of Lie groups. However, we use many results from Euclidean harmonic analysis. In fact, as we have already remarked, our aim is to develop several topics from the classical Fourier analysis in the noncommutative setup of the Heisenberg group. The reader is therefore expected to have a good foundation of Euclidean harmonic analysis. We recommend the books *An Introduction to Fourier Analysis on Euclidean Spaces* by E.M. Stein and G. Weiss and [62] of C. Sogge. We use standard notations followed in the above-mentioned books. However, we would like to warn the reader that due to the shortage of new notations, we have used the same symbol to denote the Euclidean Fourier transform as well as the group Fourier transform. Similarly, the Fourier-Weyl transform and the partial Fourier transform are denoted by the same symbol. We hope that the context will make it clear which transform is being considered.

This work is an outgrowth of the lecture notes of a course I gave at UNM, Albuquerque, during the spring of 1997. Earlier in 1994, during the Harmonic Analysis meeting in I.I.T., Mumbai, I gave a series of five lectures on the theme of harmonic analysis on the Heisenberg group. My aim had been to show, without proofs, how the standard results of Euclidean harmonic analysis look in the context of the Heisenberg group. Later, in 1996, I elaborated on some of the topics and gave a series of lectures in the I.S.I winter school held in New Delhi. Ever since I have been contemplating expanding those lectures into a monograph that could serve as a full length text for a course. This past spring, the Department of Mathematics and Statistics gave me comfortable chair to carry out the plan.

I have chosen the topics in this book according to my taste and understanding. To keep the exposition simple, some results are stated without proof. In some cases, I have sacrificed optimality for the sake of simplicity. I have indicated some conjectures and there are many open problems worthy of further investigation. I am afraid that while this work may not describe any great peak in the world of mathematics, there will be some enchanting mesas to be enjoyed. In the desert sand of these pages, I hope, the careful reader will find some wild flowers of cactus and yucca.

It is a great pleasure to express my gratitude to various people who made this monograph possible. First of all, I am grateful to Alladi Sitaram for persuading me to give lectures on the Heisenberg group in the I.S.I winter school. I thank my friends Jay Epperson and Cristina Pereyra and the students C. Dochitoiu and S. Zheng who attended my lectures with enthusiasm. The encouraging remarks of G.B. Folland and R. S. Strichartz are gratefully acknowledged.

It goes without saying that I am immensely thankful to my wife and daughters — my little ϵ and δ — for keeping me relaxed during the preparation of these notes. I am also thankful to all our Indian friends in Albuquerque who made our stay here enjoyable. Finally, I wish to thank the Indian Statistical Institute for giving me leave and the Department of Mathematics and Statistics at UNM for providing me with excellent facilities and warm hospitality.

S. Thangavelu
Albuquerque
May, 1997.

The referees made a careful and thorough study of the manuscript. I have incorporated several modifications in this final version from their suggestions for which I am grateful. I wish to thank M. Sundari for her meticulous proofreading of the manuscript and for preparing the index. It has been a pleasure working with the staff of Birkhäuser in publishing this monograph. Their kind cooperation is thankfully acknowledged.

S. Thangavelu,
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November, 1997.

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