



***Modeling and Simulation in Science, Engineering and Technology***

***Series Editor***

Nicola Bellomo  
Politecnico di Torino  
Italy

***Advisory Editorial Board***

*K.J. Bathe*  
Massachusetts Institute of Technology  
USA

*W. Kliemann*  
Iowa State University  
USA

*S. Nikitin*  
Arizona State University  
USA

*V. Protopopescu*  
CSMD  
Oak Ridge National Laboratory  
USA

*P. Degond*  
Université P. Sabatier Toulouse 3  
France

*P. Le Tallec*  
INRIA  
France

*K.R. Rajagopal*  
Texas A&M University  
USA

*Y. Sone*  
Kyoto University  
Japan

*E.S. Subuhi*  
Istanbul Technical University  
Turkey

Sergey P. Kiselev  
Evgenii V. Vorozhtsov  
Vasily M. Fomin

# Foundations of Fluid Mechanics with Applications

Problem Solving Using  
*Mathematica*<sup>®</sup>

Springer Science+Business Media, LLC

Sergey P. Kiselev  
Evgenii V. Vorozhtsov  
Vasily M. Fomin  
Institute of Theoretical and Applied Mechanics  
Russian Academy of Sciences  
Novosibirsk 630090  
Russia

**Library of Congress Cataloging-in-Publication Data**

Kiselev, S.P. (Sergey Petrovich)

Foundations of fluid mechanics with applications : problem solving  
using Mathematica / Sergey P. Kiselev, Evgenii V. Vorozhtsov, Vasily  
M. Fomin.

p. cm. (Modeling and simulation in science, engineering and  
technology)

Includes bibliographical references and index.

ISBN 978-1-4612-7198-7 ISBN 978-1-4612-1572-1 (eBook)

DOI 10.1007/978-1-4612-1572-1

1. Fluid mechanics. 2. Fluid mechanics—Data processing.  
3. Mathematica (Computer file) I. Vorozhtsov, E.V. (Evgenii  
Vasil'evich), 1946– . II. Fomin, V.M., d.f.–m.n. III. Title.  
IV. Series.

QA901.K58 1999

532'.00285'53042—dc21

99-14395  
CIP

---

AMS Subject Classifications: 76-01, 76M

---

Printed on acid-free paper.

© 1999 Springer Science+Business Media New York  
Originally published by Birkhäuser Boston in 1999  
Softcover reprint of the hardcover 1st edition 1999



All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Birkhäuser Boston, c/o Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

**ISBN 978-1-4612-7198-7**

*Mathematica*® is a registered trademark of Wolfram Research, Inc., 100 Trade Center Drive, Champaign, IL 61820-7237, USA.

Formatted from the authors' LaTeX files.

9 8 7 6 5 4 3 2 1

# Contents

<b>Preface</b>	<b>xi</b>
<b>1 Definitions of Continuum Mechanics</b>	<b>1</b>
1.1 Vectors and Tensors . . . . .	1
1.1.1 Covariant Differentiation . . . . .	5
1.1.2 The Levi–Civita Tensor . . . . .	7
1.1.3 Differential Operations . . . . .	9
1.1.4 Physical Components of Vectors and Tensors . . . . .	9
1.1.5 Eigenvalues and Eigenvectors of a Symmetric Tensor . . . . .	10
1.1.6 The Ostrogradsky–Gauss Theorem . . . . .	12
1.1.7 The Stokes Theorem . . . . .	14
1.1.8 The Weyl Formula . . . . .	15
1.2 Eulerian and Lagrangian Description of a Continuum: Strain Tensor . . . . .	24
1.2.1 Lagrangian and Eulerian Description of a Continuum . . . . .	24
1.2.2 Strain Tensor . . . . .	28
1.2.3 A Condition for Compatibility of Deformations . . . . .	35
1.2.4 Rate-of-Strain Tensor: Cauchy–Helmholtz Theorem . . . . .	37
1.3 Stress Tensor . . . . .	55
1.3.1 The Cauchy Stress Tensor in the Accompanying Coordinate System . . . . .	55
1.3.2 Piola–Kirchhoff Stress Tensors in the Reference Frame and in the Eulerian Coordinates . . . . .	59
1.3.3 Principal Values and Invariants of the Stress Tensor . . . . .	61
1.3.4 Differentiation of the Stress Tensor with Respect to Time . . . . .	63
References . . . . .	73

<b>2</b>	<b>Fundamental Principles and Laws of Continuum Mechanics</b>	<b>75</b>
2.1	Equations of Continuity, Motion, and Energy for a Continuum . . . . .	75
2.1.1	Continuity Equation . . . . .	76
2.1.2	Equations of Motion and of Momentum Moment . . . . .	78
2.1.3	The Energy Conservation Law: The First and Second Laws of Thermodynamics . . . . .	84
2.1.4	Equation of State (General Relations) . . . . .	92
2.1.5	Equations of an Ideal and Viscous, Heat-Conducting Gas . . . . .	95
2.2	The Hamilton–Ostrogradsky’s Variational Principle in Continuum Mechanics . . . . .	115
2.2.1	Euler–Lagrange Equations in Lagrangian Coordinates . . . . .	115
2.2.2	Hamilton’s Equations in Lagrangian Coordinates . . . . .	121
2.2.3	Euler–Lagrange Equations in Eulerian Coordinates and Murnaghan’s Formula . . . . .	125
2.3	Conservation Laws for Energy and Momentum in Continuum Mechanics . . . . .	135
2.3.1	Conservation Laws in Cartesian Coordinates . . . . .	135
2.3.2	Conservation Laws in an Arbitrary Coordinate System . . . . .	144
	References . . . . .	152
<b>3</b>	<b>The Features of the Solutions of Continuum Mechanics Problems</b>	<b>155</b>
3.1	Similarity and Dimension Theory in Continuum Mechanics . . . . .	155
3.2	The Characteristics of Partial Differential Equations . . . . .	163
3.3	Discontinuity Surfaces in Continuum Mechanics . . . . .	171
	References . . . . .	185
<b>4</b>	<b>Ideal Fluid</b>	<b>187</b>
4.1	Integrals of Motion Equations of Ideal Fluid and Gas . . . . .	187
4.1.1	Motion Equations in the Gromeka–Lamb Form . . . . .	188
4.1.2	The Bernoulli Integral . . . . .	188
4.1.3	The Lagrange Integral . . . . .	189
4.2	Planar Irrotational Steady Motions of an Ideal Incompressible Fluid . . . . .	193
4.2.1	The Governing Equations of Planar Flows . . . . .	193
4.2.2	The Potential Flow past the Cylinder . . . . .	202

4.2.3	The Method of Conformal Mappings . . . . .	208
4.2.4	The Problem of the Flow around a Slender Profile . . . . .	219
4.3	Axisymmetric and Three-Dimensional Potential Ideal Incompressible Fluid Flows . . . . .	223
4.3.1	Axially Symmetric Flows . . . . .	223
4.3.2	The Method of Sources and Sinks . . . . .	231
4.3.3	The Program prog4-5.nb . . . . .	233
4.3.4	The Transverse Flow around the Body of Revolution: The Program prog4-6.nb . . . . .	235
4.4	Nonstationary Motion of a Solid in the Fluid . . . . .	242
4.4.1	Formulation of a Problem on Nonstationary Body Motion in Ideal Fluid . . . . .	242
4.4.2	The Hydrodynamic Reactions at the Body Motion . . . . .	244
4.4.3	Equations of Solid Motion in a Fluid under the Action of Given Forces . . . . .	247
4.5	Vortical Motions of Ideal Fluid . . . . .	250
4.5.1	The Theorems of Thomson, Lagrange, and Helmholtz . . . . .	250
4.5.2	Motion Equations in Friedmann's Form . . . . .	257
4.5.3	The Biot–Savart Formulas and the Straight Vortex Filament . . . . .	258
	References . . . . .	265
<b>5</b>	<b>Viscous Fluid</b>	<b>267</b>
5.1	General Equations of Viscous Incompressible Fluid . . . . .	268
5.1.1	The Navier–Stokes Equations . . . . .	268
5.1.2	Formulation of Problems for the System of the Navier–Stokes Equations . . . . .	275
5.2	Viscous Fluid Flows at Small Reynolds Numbers . . . . .	276
5.2.1	Exact Solutions of the System of Equations for a Viscous Fluid . . . . .	277
5.2.2	Viscous Fluid Motion between Two Rotating Coaxial Cylinders . . . . .	280
5.2.3	The Viscous Incompressible Fluid Flow around a Sphere at Small Reynolds Numbers . . . . .	282
5.3	Viscous Fluid Flows at Large Reynolds Numbers . . . . .	287
5.3.1	Prandtl's Theory of Boundary Layers . . . . .	288
5.3.2	Boundary Layer of a Flat Plate . . . . .	293
5.4	Turbulent Fluid Flows . . . . .	298
5.4.1	Basic Properties of Turbulent Flows . . . . .	298
5.4.2	Laminar Flow Stability and Transition to Turbulence . . . . .	300

5.4.3	Turbulent Fluid Flow . . . . .	302
	References . . . . .	309
<b>6</b>	<b>Gas Dynamics</b>	<b>311</b>
6.1	One-Dimensional Stationary Gas Flows . . . . .	311
6.1.1	Governing Equations for Quasi-One-Dimensional Gas Flow . . . . .	311
6.1.2	Gas Motion in a Variable Section Duct: Elementary Theory of the Laval Nozzle . . . . .	313
6.1.3	Planar Shock Wave in Ideal Gas . . . . .	321
6.1.4	Shock Wave Structure in Gas . . . . .	329
6.2	Nonstationary One-Dimensional Flows of Ideal Gas . . . . .	334
6.2.1	Planar Isentropic Waves . . . . .	334
6.2.2	Gradient Catastrophe and Shock Wave Formation . . . . .	342
6.3	Planar Irrotational Ideal Gas Motion (Linear Approximation) . . . . .	346
6.3.1	Governing Equations and Their Linearization . . . . .	346
6.3.2	The Problem of the Flow around a Slender Profile . . . . .	348
6.4	Planar Irrotational Stationary Ideal Gas Flow (General Case) . . . . .	354
6.4.1	Characteristics of Stationary Irrotational Flows of Ideal Gas, Simple Wave: The Prandtl–Meyer Flow . . . . .	355
6.4.2	Chaplygin’s Equations and Method . . . . .	366
6.4.3	Oblique Shock Waves . . . . .	377
6.4.4	Interference of Stationary Shock Waves . . . . .	382
6.5	The Fundamentals of the Gasdynamic Design Technology . . . . .	386
6.5.1	The Basic Algorithm . . . . .	387
6.5.2	The Superposition Procedure . . . . .	391
6.5.3	The Complement Procedure . . . . .	395
	References . . . . .	399
<b>7</b>	<b>Multiphase Media</b>	<b>401</b>
7.1	Mathematical Models of Multiphase Media . . . . .	403
7.1.1	General Equations of the Mechanics of Multiphase Media . . . . .	403
7.1.2	Equations of a Two-Phase Medium of the Type of Gas–Solid Particles . . . . .	407
7.1.3	Equations of a Bicomponent Medium of Gas Mixture Type . . . . .	415



7.2	Correctness of the Cauchy Problem: Relations at Discontinuities in Multiphase Media . . . . .	417
7.2.1	The Characteristics of a System of Equations for Gas-Particle Mixtures and Correctness of the Cauchy Problem . . . . .	417
7.2.2	Jump Relations . . . . .	431
7.3	Quasi-One-Dimensional Flows of a Gas-Particle Mixture in Laval Nozzles . . . . .	442
7.3.1	The Equations of the Quasi-One-Dimensional Flow of a Gas-Particle Mixture . . . . .	442
7.3.2	The Flow of a Gas-Particle Mixture in the Laval Nozzle with Small Velocity and Temperature Lags of Particles . . . . .	447
7.4	The Continual-Discrete Model and Caustics in the Pseudogas of Particles . . . . .	456
7.4.1	The Equations of the Continual-Discrete Model of a Gas-Particle Mixture at a Small Volume Concentration of Particles . . . . .	456
7.4.2	Investigation of Caustics in the Pseudogas of Particles . . . . .	460
7.5	Nonstationary Processes in Gas-Particle Mixtures . . . . .	471
7.5.1	Interaction of a Shock Wave with a Cloud of Particles . . . . .	471
7.5.2	Acoustic Approximation in the Problem of Shock Wave Interaction with a Particle's Cloud at a Small Volume Concentration . . . . .	478
7.6	The Flows of Heterogeneous Media without Regard for Inertial Effects . . . . .	486
7.6.1	The Brownian Motion of Particles in a Fluid . . . . .	486
7.6.2	Fluid Filtration in a Porous Medium . . . . .	493
7.7	Wave Processes in Bubbly Liquids . . . . .	500
7.7.1	Equations of the Motion of a Bubbly Liquid . . . . .	500
7.7.2	Equations for Weak Nonlinear Disturbances in Bubbly Liquids . . . . .	507
7.7.3	Progressive, Weak Nonlinear Waves in Bubbly Liquids . . . . .	512
	References . . . . .	522
	<b>Appendix A: <i>Mathematica</i> Functions</b>	<b>526</b>
	<b>Appendix B: Glossary of Programs</b>	<b>550</b>
	<b>Index</b>	<b>565</b>

# Preface

Fluid mechanics (FM) is a branch of science dealing with the investigation of flows of continua under the action of external forces. The fundamentals of FM were laid in the works of the famous scientists, such as L. Euler, M.V. Lomonosov, D. Bernoulli, J.L. Lagrange, A. Cauchy, L. Navier, S.D. Poisson, and other classics of science. Fluid mechanics underwent a rapid development during the past two centuries, and it now includes, along with the above branches, aerodynamics, hydrodynamics, rarefied gas dynamics, mechanics of multiphase and reactive media, etc.

The FM application domains were expanded, and new investigation methods were developed. Certain concepts introduced by the classics of science, however, are still of primary importance and will apparently be of importance in the future. The Lagrangian and Eulerian descriptions of a continuum, tensors of strains and stresses, conservation laws for mass, momentum, moment of momentum, and energy are the examples of such concepts and results. This list should be augmented by the first and second laws of thermodynamics, which determine the character and direction of processes at a given point of a continuum. The availability of the conservation laws is conditioned by the homogeneity and isotropicity properties of the Euclidean space, and the form of these laws is related to the Newton's laws. The laws of thermodynamics have their foundation in the statistical physics. These concepts and laws are insufficient for the description of continuum motion, however, they should be augmented by the equations of state or, as one says sometimes, by the closure relationships, which relate the tensors of strains and stresses to their derivatives. Each continuum possesses its own equations of state, which are found by processing the experimental data or by solving the corresponding problems from the statistical physics.

The concept of continuum does not contain in itself the information about the discrete character of substance at the molecular level, all this information is laid in thermodynamics laws and closure relationships (equations of state). It follows from here that the FM is no closed field of knowledge; it has a tight interaction with other branches of physics and mathematics.

The teaching of FM has its remarkable traditions laid in the lecture

courses of L.D. Landau and E.M. Lifschitz, L.I. Sedov, N.I. Kochin, L. Prandtl, G. Batchelor, W. Prager, P. Germain, and other authors. New interesting results have appeared in FM after the publication of these lecture courses, however, and there is now a need in presenting them in the educational literature, so that students can be rapidly introduced into the scope of present-day FM problems and methods.

The development of computer algebra and of a powerful universal software system *Mathematica* has led to the fact that the task of the FM presentation with the use of the *Mathematica* system has become topical. The present authors have undertaken an attempt in this book at solving this task. A large number of programs are presented that have enabled us to perform in the process of material presentation both analytical and numeric computations with the aid of a personal computer. All *Mathematica 3.0* programs are stored on the Birkhäuser server. The url address of which is as follows (see Appendix B for further details of our programs):

<http://www.birkhauser.com/book/isbn/0-8176-3995-0>

In addition, we aimed at taking into account the international character of science. Therefore, we considered it necessary to familiarize Western readers in more detail with the achievements of the Russian scientists in the field of fluid mechanics.

The present book has been written on the basis of the lecture courses, which were presented by the authors during the past few years at the Novosibirsk State Technical University and at the Novosibirsk State University. The lectures were intended for graduate and postgraduate students that have already attended the introductory lecture courses in FM. At the same time, this book gives all of the necessary FM concepts and the derivations of all formulas are presented. There are also a large number of problems with the solutions. All of these features enable one to use this book both for an initial and a deeper study of FM.

Note that, although we present FM as a theoretical discipline, its development is closely related to the experiment and the practical needs of industry. The role of experiment in the formation of the basic concepts and closing relationships of FM is very significant. After these concepts have been established, however, the deductive methods of mathematics take on deciding significance.

The book consists of seven chapters. Chapter 1 presents the basic concepts of continua: the Lagrangian and Eulerian description, tensors of strains and stresses, equations of continuity, momentum, and energy. We use throughout the chapter a tensor invariant form, which does not depend on the choice of coordinate system. Therefore, the presentation of the basic FM concepts is preceded by a brief introduction in tensor analysis. The basic definitions of the tensor analysis, which are used

in the following, are briefly introduced. From the very beginning, the strains are not assumed to be small; therefore, various tensors of strains and stresses are introduced that are related to the initial and current configurations at the Lagrangian description and in a fixed Eulerian frame.

In Chapter 2, we present the derivation of the differential equations for continuity, energy, and motion. The concept of local thermodynamic equilibrium as well as the thermodynamics laws enable us to indicate the general form of the closure relationships for the FM governing equations. We formulate the Hamilton–Ostrogradsky variational principle, which enables us, on the one hand, to find the FM motion equations and, on the other hand, to establish a relation between the integral conservation laws for energy and momentum and the isotropicity and homogeneity properties of space and time.

In Chapter 3, the fundamentals of the similarity and dimension theory as well as the mathematical methods for studying the weak discontinuities (the characteristics) and strong discontinuities in fluid mechanics are presented.

Chapter 4 is devoted to the fundamentals of the theory of dynamics of ideal incompressible fluid. For the ideal fluid, we derive the Bernoulli and Thomson integrals and consider the planar and axisymmetric irrotational flows. We also study the general properties of the vortex flows. We further consider in detail the methods for the solution of problems on the ideal fluid flow around planar and axisymmetric bodies.

Chapter 5 deals with the viscous fluid flows. The Navier–Stokes equations are derived, and a number of the solutions of these equations are obtained at small Reynolds numbers. The basics of the Prandtl’s boundary layer theory are presented. Some approaches to the description of turbulent fluid flows are considered.

Chapter 6 is devoted to the gas dynamics of ideal and viscous compressible gases. We present the theory of the Laval nozzle, shock waves, and Riemann waves. We also give the solution of a problem on shock wave structure in a viscous gas. The Chaplygin’s theory for the transformation of gas dynamics equations to the hodograph plane is presented in sufficient detail.

Chapter 7 is devoted to a new FM branch, the mechanics of multiphase heterogeneous media. This branch of mechanics has appeared during the past 20–30 years, and it now enjoys a period of intense development. One can speak today about the fact that the general approaches have been developed, which are applicable to the description of an arbitrary multiphase medium. Various methods of averaging belong to them, which enable one to go over to an averaged description, as well as the idea of interpenetrating continua, each of which refers to a corresponding phase.

A mathematical model of the gas–particle flow was historically the first model describing multiphase flows. This was related to the practical applications in the field of two-phase gas dynamics of Laval nozzles and the flow of dusty gas around the flying vehicles. The developed methods were then used in the models of bubbly fluid flows, porous materials, and gas mixtures. For this reason, sufficient attention is paid here to the model of gas–particle flow. It should be noted that there are at present in the literature a number of monographs devoted to the mechanics of multiphase media. These monographs are cited in the list of References. Our presentation is nevertheless different from them, since a number of the results presented in our book belongs to the present authors. This refers to the theory of thin discontinuities in the gas–particle mixtures, which carry a finite surface mass; the continual/discrete model, which enables one to model the flows with the intersection of particles trajectories; the theory of caustics in the pseudogas of particles, which gives a limitation for the total number of particles lying on a caustic as well as the condition for its formation; the theory of shock wave interaction with a particle’s cloud within the framework of which an explanation is given for the formation of a collective shock wave upstream of the particle’s cloud at a volume concentration of particles of the order of several percents.

A triple numbering of formulas is used in the book. The first number indicates the chapter number, the second number is the section number, and the third number is the formula number in the section.

The authors hope that this book will be useful for students of universities and higher technical colleges as well as for specialists working in the field of FM.

The authors express their deep gratitude to professional colleagues whose discussions contributed to the elucidation of many complex questions of FM.