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Albrecht Böttcher Bernd Silbermann

Introduction to Large Truncated Toeplitz Matrices

With 62 Figures

Springer

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Preface

Toeplitz matrices have been enjoying immense popularity for many decades. They are easily defined (as matrices constant along the parallels to the main diagonal), they emerge in a variety of problems of quite different natures, they cause interesting and difficult questions, and they usually lead to beautiful results. On the one hand, Toeplitz matrices are easy enough to serve as ideal illustrations of various abstract results and methods of linear algebra and functional analysis, and on the other hand, they are sufficiently nontrivial and therefore have the potential to create new concepts, techniques, insights, and, of course, to raise new questions.

Figuratively speaking, the theory of Toeplitz matrices has grown to a grandiose city. Specialists know the parts of this city well and are about to reconstruct and expand it, but beginners and amateurs often have problems with finding their way in the labyrinth. This book is addressed to the latter group of people. It is intended as a guide to three main roads of this city, whose names are:

pseudospectra,
singular values,
eigenvalues,

and the book also contains glimpses at several side-streets.

We consider large finite Toeplitz matrices as truncations of infinite Toeplitz matrices and hence, we study properties of an individual large finite Toeplitz matrix by embedding it into the sequence of the truncations (finite sections) of an infinite Toeplitz matrix.

The three roads mentioned start at a place that bears the name

stability.

Given an infinite Toeplitz matrix A, let $\{A_n\}_{n=1}^{\infty}$ stand for the sequence of its $n \times n$ truncations. The central problem of the entire theory is as follows: if A induces an invertible operator, are the finite sections A_n invertible for all sufficiently large n, say for $n \geq n_0$, and are the norms of the inverses, $\|A_n^{-1}\|$, bounded from above by a finite constant independent of $n \geq n_0$? If this is the case, the sequence $\{A_n\}$ is said to be stable. Properties of infinite Toeplitz matrices (including invertibility criteria) are studied in Chapter 1 and the stability of sequences of truncated Toeplitz matrices is the topic of Chapter 2.

Chapters 3,4, and 5 deal with pseudospectra, singular values, and eigenvalues of the truncated matrices A_n, respectively. The investigation of the pseudospectra of A_n is heavily based on the ability of computing the limit of $\|A_n^{-1}\|$ as n goes to infinity. The computation of this limit is in turn a nice application of the theory of C^*-algebras. The asymptotic distribution of the singular values of A_n is intimately tied in with asymptotic Moore-Penrose inversion of the matrices A_n. Finally, the main results on the asymptotic behavior of the eigenvalues of A_n are all derived from asymptotic formulas for the traces and determinants of Toeplitz-like matrices. Thus, we could also name our three main roads:

condition numbers,
Moore-Penrose inversion,
traces and determinants,

respectively.

The matrices one encounters nowadays in applications are often not Toeplitz matrices but block Toeplitz matrices. Many results on Toeplitz matrices can be extended to block Toeplitz matrices, although this is usually a hard job. In accordance with the purpose of this text, we focus our attention on scalar Toeplitz matrices. However, in Chapter 6 we describe some of the phenomena caused by block Toeplitz matrices and cite several results, referring for proofs to the literature.

In Chapter 7, we exhibit some results on Toeplitz operators on Banach spaces and acquaint the reader with certain techniques employed in this field. We remark that the Banach space theory of Toeplitz operators with piecewise continuous symbols is more beautiful than the corresponding Hilbert space theory! Moreover, we will exemplify in Chapter 7 that there are Hilbert space results which can be most easily understood by passing to Banach spaces.

We suppose that knowledge of the basic facts of functional analysis in conjunction with some patience and persistence should suffice to enable a reading of the bulk of the text. Of course, the sights along the three roads mentioned will represent our taste, and some important topics are not treated at all. We would be happy if we could nevertheless convey to the reader an idea of what is known and of what is going on in the extensive and beautiful field of large Toeplitz matrices.

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Chemnitz, December 1997

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