Universitext

Springer Science+Business Media, LLC

Universitext

Editors (North America): S. Axler, F.W. Gehring, and K.A. Ribet

Aksoy/Khamsi: Nonstandard Methods in Fixed Point Theory
Andersson: Topics in Complex Analysis
Aupetit: A Primer on Spectral Theory
Balakrishnan/Ranganathan: A Textbook of Graph Theory
Balser: Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations
Bapat: Linear Algebra and Linear Models (2nd ed.)
Berberian: Fundamentals of Real Analysis
Booss/Bleecker: Topology and Analysis
Borkar: Probability Theory: An Advanced Course
Böttcher/Silbermann: Introduction to Large Truncated Toeplitz Matrices
Carleson/Gamelin: Complex Dynamics
Cecil: Lie Sphere Geometry: With Applications to Submanifolds
Chae: Lebesgue Integration (2nd ed.)
Charlap: Bieberbach Groups and Flat Manifolds
Chern: Complex Manifolds Without Potential Theory
Cohn: A Classical Invitation to Algebraic Numbers and Class Fields
Curtis: Abstract Linear Algebra
Curtis: Matrix Groups
DiBenedetto: Degenerate Parabolic Equations
Dimca: Singularities and Topology of Hypersurfaces
Edwards: A Formal Background to Mathematics I a/b
Edwards: A Formal Background to Mathematics II a/b
Foulds: Graph Theory Applications
Friedman: Algebraic Surfaces and Holomorphic Vector Bundles
Fuhrmann: A Polynomial Approach to Linear Algebra
Gardiner: A First Course in Group Theory
Gårding/Tambour: Algebra for Computer Science
Goldblatt: Orthogonality and Spacetime Geometry
Gustafson/Rao: Numerical Range: The Field of Values of Linear Operators and Matrices
Hahn: Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups
Holmgren: A First Course in Discrete Dynamical Systems
Howe/Tan: Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathrm{R})$
Howes: Modern Analysis and Topology
Hsieh/Sibuya: Basic Theory of Ordinary Differential Equations
Humi/Miller: Second Course in Ordinary Differential Equations
Hurwitz/Kritikos: Lectures on Number Theory
Jennings: Modern Geometry with Applications
Jones/Morris/Pearson: Abstract Algebra and Famous Impossibilities
Kannan/Krueger: Advanced Analysis
Kelly/Matthews: The Non-Euclidean Hyperbolic Plane
Kostrikin: Introduction to Algebra
Luecking/Rubel: Complex Analysis: A Functional Analysis Approach
MacLane/Moerdijk: Sheaves in Geometry and Logic
Marcus: Number Fields
McCarthy: Introduction to Arithmetical Functions

(continued after index)

Heydar Radjavi Peter Rosenthal

Simultaneous Triangularization

Springer

Heydar Radjavi
Department of Mathematics
Dalhousie University
Halifax, Nova Scotia B3H 3J5
Canada

Peter Rosenthal
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1
Canada

Mathematics Subject Classification (1991): 15-02, 15A21, 15A30, 15A48, 47-02, 47A15, 47A65, 47C05, 47D03

Library of Congress Cataloging-in-Publication Data
Radjavi, Heydar.
Simultaneous triangularization / Heydar Radjavi, Peter Rosenthal.
p. cm. — (Universitext)
Includes bibliographical references and index.
ISBN 978-0-387-98466-7 ISBN 978-1-4612-1200-3 (eBook)
DOI 10.1007/978-1-4612-1200-3
1. Triangularization (Mathematics) I. Rosenthal, Peter, 1941–
II. Title. III. Series.
QA197.R33 2000
512.9'434—dc21 99-23772

Printed on acid-free paper.

Production managed by Allan Abrams; manufacturing supervised by Jeffrey Taub.
Photocomposed copy prepared from the authors' AMS-TeX files.

9 8 7 6 5 4 3 2 1

ISBN 978-0-387-98466-7

To

Ursula, Marjan,
Shirin, Carrie,

Carol, Alan,
Jeffrey, Michael,
Daniel, Esther,
Laurel, Jeremy,
Aaron, Margaret

PREFACE

A matrix over the field of complex numbers (or over any other algebraically closed field) is similar to a matrix in upper triangular form. For individual matrices, this is not as important as the Jordan canonical form (partially because it is not canonical), but it is nonetheless useful in many situations. In particular, the eigenvalues of a matrix in upper triangular form are visible: They are just the elements on the main diagonal of the matrix. Moreover, as is illustrated throughout this book, many collections of matrices can be simultaneously put in upper triangular form, whereas it is very rare that two matrices have "simultaneous Jordan forms."

When are two matrices simultaneously similar to matrices in upper triangular form? Equivalently, given two linear transformations on a finite-dimensional complex vector space, when is there a basis for the space with respect to which the matrices of both transformations are upper triangular? More generally, when is a collection of linear transformations "simultaneously triangularizable"?

It is easily shown that commutative sets of matrices are simultaneously triangularizable. Simultaneous triangularizability can be regarded as a kind of generalized commutativity, and implies certain consequences of commutativity. In particular, a set $\{A_1, A_2, \ldots, A_m\}$ of linear transformations is simultaneously triangularizable if and only if, for every polynomial p, each eigenvalue of $p(A_1, A_2, \ldots, A_m)$ has the form $p(\lambda_1, \lambda_2, \ldots, \lambda_m)$ where λ_j is an eigenvalue of A_j for each j. Also, simultaneous triangularizability is equivalent to the existence of simultaneous similarities that are close to commuting transformations.

There are many beautiful classical theorems, associated with names such as Engel, McCoy, Levitzki, Kolchin, and Kaplansky, giving sufficient conditions that collections of linear transformations be simultaneously triangularizable. There are also a number of more recent results by many researchers. Some of the work on triangularizability intersects other areas of linear algebra. For example, triangularization theorems for collections of nonnegative matrices relate to the Perron-Frobenius theory. Triangularizability is also linked to various partial spectral mapping theorems, and to properties of spectral radii and traces.

Beginning around 1980, the theory has been extended to operators on infinite-dimensional Banach spaces. A collection of bounded linear operators is said to be simultaneously triangularizable (or, simply, triangularizable) if there is a maximal chain of subspaces, each of which is left invariant by all the operators in the collection. Many of the finite-dimensional results have satisfactory infinite-dimensional generalizations to collections of compact operators; the basis for such theorems is the lemma established by Lomonosov in his famous paper of 1973, together with Ringrose's Theorem on computing the spectrum of a compact operator from a "triangular" representation. These fundamental results have been supplemented by Turovskii's just-discovered (1998) extension of Lomonosov's work from algebras to semigroups.

There is now a great deal known about triangularizability of collections of compact operators. In particular, simultaneous triangularizability is equivalent to a spectral mapping property for collections of compact operators. As in the finite-dimensional situation, triangularizability generalizes commutativity, and is implied by several other kinds of generalized commutativity. Also, there are many sufficient conditions that collections of compact operators be triangularizable.

Very few of the above results generalize further to arbitrary bounded operators: There are even collections containing just a single operator that have only the trivial invariant subspaces (as shown by Per Enflo), and which are therefore very far from triangularizable. It is not known whether every operator on Hilbert space has a nontrivial invariant subspace, and therefore it is not known whether every operator on Hilbert space is triangularizable. However, even in the Hilbert space case there are counterexamples to most of the natural generalizations of the finite-dimensional results. On the other hand, there are several affirmative results.

In this book we have attempted to give a fairly complete treatment of the classical and recent results in both the finite- and infinite-dimensional settings. We have reworked much of the material to provide a more cohesive and readable treatment than can be obtained by simply taking the union of the published research papers. Moreover, we aspired to make the exposition as elementary and self-contained as possible. In addition, we have included a number of new results.

We hope that this book will be found useful by graduate students and mathematicians working in or contemplating work in either the finite- or infinite-dimensional aspects of this topic. We also hope that making these results more easily available will increase the applications of simultaneous triangularizability to other areas, such as representations of groups and semigroups. The finite-dimensional results are treated independently in the first five chapters; readers who are not interested in operators on Banach spaces may restrict their attention to these chapters. However, we have written the infinite-dimensional sections with a view to making them accessible to those with minimal backgrounds in functional analysis. In particular, Chapter 6 includes a discussion of the basic material required; we hope this encourages all readers to at least peruse the latter part of the book as well. On the other hand, those primarily interested in operator theory could begin with Chapter 7 and read only those earlier sections that are directly referred to thereafter. However, we would suggest that such readers would benefit by at least skimming the earlier chapters first.

In addition, we have written this book so as to make it suitable for students to read as a text. In fact, a very preliminary version formed the basis for graduate courses at Dalhousie University and the University of Toronto. Parts of the book might be used in various courses. The only prerequisite for the first five chapters is a solid course in linear algebra, and all that is required for the balance is an introductory course in functional analysis. Instructors who do not want to spend an entire semester on simultaneous triangularization might use parts of the book as a source of topics in courses covering other material.

Each chapter ends with a section entitled "Notes and Remarks" in which we attempt to outline the development of the material and to discuss other interesting results that could not be included in the main text. Some of these discussions and references could provide direction for reading projects for students.

We use the following scheme for numbering definitions, lemmas, theorems and corollaries throughout the book: chapter.section.number. For example, 7.2.3 is the third numbered item in Section 2 of Chapter 7.

We are grateful to several mathematicians who made suggestions and caught errors (we hope they caught most of them) after reading preliminary versions of the manuscript, especially Marjeta Kramar, Bill Longstaff, Mitja Mastnak, M. H. Shirdarreh, Reza Yahaghi, and Yong Zhong. We are particularly grateful to Ruben Martinez, who found a large number of mistakes, made several excellent suggestions, and was of great assistance in compiling the references and indices. We are also grateful to the three people who did a wonderful job of transforming our sloppy scrawls into beautiful type: Maria Fe Elder, Lucile Lo, and Karin Smith. After this process should have ended, we made an almost infinite number of additions and corrections, which Lucile Lo handled with infinite patience and skill.

The older we grow, the more we appreciate those who taught and encouraged us when we were young, especially Ali Afzalipour, Arthur B. Brown, Robert Cameron, Chandler Davis, Taghi Fatemi, Israel Halperin, Gerhard Kalish, Mohammad Ali Nourghalitchi, Harold Rosenthal, Allen Shields, Manoutchehr Vessal, and Leo Zippin. We are particularly grateful to Paul Halmos for providing inspiration and guidance throughout our careers.

Heydar Radjavi
Dalhousie University

Peter Rosenthal
University of Toronto

CONTENTS