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*(continued after index)*

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# Simultaneous Triangularization



Springer

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## PREFACE

A matrix over the field of complex numbers (or over any other algebraically closed field) is similar to a matrix in upper triangular form. For individual matrices, this is not as important as the Jordan canonical form (partially because it is not canonical), but it is nonetheless useful in many situations. In particular, the eigenvalues of a matrix in upper triangular form are visible: They are just the elements on the main diagonal of the matrix. Moreover, as is illustrated throughout this book, many collections of matrices can be simultaneously put in upper triangular form, whereas it is very rare that two matrices have “simultaneous Jordan forms.”

When are two matrices simultaneously similar to matrices in upper triangular form? Equivalently, given two linear transformations on a finite-dimensional complex vector space, when is there a basis for the space with respect to which the matrices of both transformations are upper triangular? More generally, when is a collection of linear transformations “simultaneously triangularizable”?

It is easily shown that commutative sets of matrices are simultaneously triangularizable. Simultaneous triangularizability can be regarded as a kind of generalized commutativity, and implies certain consequences of commutativity. In particular, a set  $\{A_1, A_2, \dots, A_m\}$  of linear transformations is simultaneously triangularizable if and only if, for every polynomial  $p$ , each eigenvalue of  $p(A_1, A_2, \dots, A_m)$  has the form  $p(\lambda_1, \lambda_2, \dots, \lambda_m)$  where  $\lambda_j$  is an eigenvalue of  $A_j$  for each  $j$ . Also, simultaneous triangularizability is equivalent to the existence of simultaneous similarities that are close to commuting transformations.

There are many beautiful classical theorems, associated with names such as Engel, McCoy, Levitzki, Kolchin, and Kaplansky, giving sufficient conditions that collections of linear transformations be simultaneously triangularizable. There are also a number of more recent results by many researchers. Some of the work on triangularizability intersects other areas of linear algebra. For example, triangularization theorems for collections of nonnegative matrices relate to the Perron-Frobenius theory. Triangularizability is also linked to various partial spectral mapping theorems, and to properties of spectral radii and traces.

Beginning around 1980, the theory has been extended to operators on infinite-dimensional Banach spaces. A collection of bounded linear operators is said to be simultaneously triangularizable (or, simply, triangularizable) if there is a maximal chain of subspaces, each of which is left invariant by all the operators in the collection. Many of the finite-dimensional results have satisfactory infinite-dimensional generalizations to collections of compact operators; the basis for such theorems is the lemma established by Lomonosov in his famous paper of 1973, together with Ringrose’s Theorem on computing the spectrum of a compact operator from a “triangular” representation. These fundamental results have been supplemented

by Turovskii's just-discovered (1998) extension of Lomonosov's work from algebras to semigroups.

There is now a great deal known about triangularizability of collections of compact operators. In particular, simultaneous triangularizability is equivalent to a spectral mapping property for collections of compact operators. As in the finite-dimensional situation, triangularizability generalizes commutativity, and is implied by several other kinds of generalized commutativity. Also, there are many sufficient conditions that collections of compact operators be triangularizable.

Very few of the above results generalize further to arbitrary bounded operators: There are even collections containing just a single operator that have only the trivial invariant subspaces (as shown by Per Enflo), and which are therefore very far from triangularizable. It is not known whether every operator on Hilbert space has a nontrivial invariant subspace, and therefore it is not known whether every operator on Hilbert space is triangularizable. However, even in the Hilbert space case there are counterexamples to most of the natural generalizations of the finite-dimensional results. On the other hand, there are several affirmative results.

In this book we have attempted to give a fairly complete treatment of the classical and recent results in both the finite- and infinite-dimensional settings. We have reworked much of the material to provide a more cohesive and readable treatment than can be obtained by simply taking the union of the published research papers. Moreover, we aspired to make the exposition as elementary and self-contained as possible. In addition, we have included a number of new results.

We hope that this book will be found useful by graduate students and mathematicians working in or contemplating work in either the finite- or infinite-dimensional aspects of this topic. We also hope that making these results more easily available will increase the applications of simultaneous triangularizability to other areas, such as representations of groups and semigroups. The finite-dimensional results are treated independently in the first five chapters; readers who are not interested in operators on Banach spaces may restrict their attention to these chapters. However, we have written the infinite-dimensional sections with a view to making them accessible to those with minimal backgrounds in functional analysis. In particular, Chapter 6 includes a discussion of the basic material required; we hope this encourages all readers to at least peruse the latter part of the book as well. On the other hand, those primarily interested in operator theory could begin with Chapter 7 and read only those earlier sections that are directly referred to thereafter. However, we would suggest that such readers would benefit by at least skimming the earlier chapters first.

In addition, we have written this book so as to make it suitable for students to read as a text. In fact, a very preliminary version formed the basis for graduate courses at Dalhousie University and the University of Toronto. Parts of the book might be used in various courses. The only

prerequisite for the first five chapters is a solid course in linear algebra, and all that is required for the balance is an introductory course in functional analysis. Instructors who do not want to spend an entire semester on simultaneous triangularization might use parts of the book as a source of topics in courses covering other material.

Each chapter ends with a section entitled “Notes and Remarks” in which we attempt to outline the development of the material and to discuss other interesting results that could not be included in the main text. Some of these discussions and references could provide direction for reading projects for students.

We use the following scheme for numbering definitions, lemmas, theorems and corollaries throughout the book: chapter.section.number. For example, 7.2.3 is the third numbered item in Section 2 of Chapter 7.

We are grateful to several mathematicians who made suggestions and caught errors (we hope they caught most of them) after reading preliminary versions of the manuscript, especially Marjeta Kramar, Bill Longstaff, Mitja Mastnak, M. H. Shirdarreh, Reza Yahaghi, and Yong Zhong. We are particularly grateful to Ruben Martinez, who found a large number of mistakes, made several excellent suggestions, and was of great assistance in compiling the references and indices. We are also grateful to the three people who did a wonderful job of transforming our sloppy scrawls into beautiful type: Maria Fe Elder, Lucile Lo, and Karin Smith. After this process should have ended, we made an almost infinite number of additions and corrections, which Lucile Lo handled with infinite patience and skill.

The older we grow, the more we appreciate those who taught and encouraged us when we were young, especially Ali Afzalipour, Arthur B. Brown, Robert Cameron, Chandler Davis, Taghi Fatemi, Israel Halperin, Gerhard Kalish, Mohammad Ali Nourghalitchi, Harold Rosenthal, Allen Shields, Manoutchehr Vessal, and Leo Zippin. We are particularly grateful to Paul Halmos for providing inspiration and guidance throughout our careers.

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## CONTENTS

<b>Preface</b>	vii
<b>Chapter One: Algebras of Matrices</b>	1
1.1 The Triangularization Lemma	1
1.2 Burnside's Theorem	4
1.3 Triangularizability of Algebras of Matrices	6
1.4 Triangularization and the Radical	10
1.5 Block Triangularization and Characterizations of Triangularizability	12
1.6 Approximate Commutativity	17
1.7 Nonassociative Algebras	21
1.8 Notes and Remarks	25
<b>Chapter Two: Semigroups of Matrices</b>	27
2.1 Basic Definitions and Propositions	27
2.2 Permutable Trace	33
2.3 Zero-One Spectra	36
2.4 Notes and Remarks	41
<b>Chapter Three: Spectral Conditions on Semigroups</b>	43
3.1 Reduction to the Field of Complex Numbers	43
3.2 Permutable Spectrum	51
3.3 Submultiplicative Spectrum	55
3.4 Conditions on Spectral Radius	62
3.5 The Dominance Condition on Spectra	70
3.6 Notes and Remarks	73
<b>Chapter Four: Finiteness Lemmas and Further Spectral Conditions</b>	75
4.1 Reductions to Finite Semigroups	75
4.2 Subadditive and Sublinear Spectra	80
4.3 Further Multiplicative Conditions on Spectra	89
4.4 Polynomial Conditions on Spectra	93
4.5 Notes and Remarks	102
<b>Chapter Five: Semigroups of Nonnegative Matrices</b>	104
5.1 Decomposability	104
5.2 Indecomposable Semigroups	115
5.3 Connections with Reducibility	127
5.4 Notes and Remarks	129
<b>Chapter Six: Compact Operators and Invariant Subspaces</b>	130
6.1 Operators on Banach Spaces	130

6.2 Compact Operators	133
6.3 Invariant Subspaces for Compact Operators	135
6.4 The Riesz Decomposition of Compact Operators	138
6.5 Trace-Class Operators on Hilbert Space	142
6.6 Notes and Remarks	149
<b>Chapter Seven: Algebras of Compact Operators</b>	<b>151</b>
7.1 The Definition of Triangularizability	151
7.2 Spectra from Triangular Forms	155
7.3 Lomonosov's Lemma and McCoy's Theorem	162
7.4 Transitive Algebras	167
7.5 Block Triangularization and Applications	173
7.6 Approximate Commutativity	183
7.7 Notes and Remarks	189
<b>Chapter Eight: Semigroups of Compact Operators</b>	<b>193</b>
8.1 Quasinilpotent Compact Operators	193
8.2 A General Approach	199
8.3 Permutability and Submultiplicativity of Spectra	205
8.4 Subadditivity and Sublinearity of Spectra	209
8.5 Polynomial Conditions on Spectra	213
8.6 Conditions on Spectral Radius and Trace	216
8.7 Nonnegative Operators	223
8.8 Notes and Remarks	241
<b>Chapter Nine: Bounded Operators</b>	<b>244</b>
9.1 Collections of Nilpotent Operators	244
9.2 Commutators of Rank One	250
9.3 Bands	262
9.4 Nonnegative Operators	272
9.5 Notes and Remarks	281
<b>References</b>	<b>284</b>
<b>Notation Index</b>	<b>307</b>
<b>Author Index</b>	<b>309</b>
<b>Subject Index</b>	<b>315</b>