Part IV
Applications to Partial Differential Equations

The purpose of this Part IV is to discuss mainly

Wave Equation: $Z_{\lambda\lambda}(\lambda,t) = \alpha^2 Z_{tt}(\lambda,t)$,

Telegraph Equation: $Z_{\lambda\lambda}(\lambda,t) = Z_{tt}(\lambda,t) + 2\kappa Z_t(\lambda,t) + \gamma Z(\lambda,t)$,

Heat Equation: $Z_{\lambda\lambda}(\lambda,t) = \alpha^2 Z_t(\lambda,t)$,

by converting these equations into hyperfunction equations.

The wave equation shall be discussed in the domain $\lambda_1 \leqq \lambda \leqq \lambda_2$, $t \geqq 0$. By making use of (5.6), we obtain

$$s^2 Z(\lambda,t) = Z_{tt}(\lambda,t) + Z_t(\lambda,0) + sZ(\lambda,0),$$

so that the initial condition with respect to t will be

$$Z(\lambda,0) = \phi(\lambda), \quad Z_t(\lambda,0) = \Psi(\lambda).$$

Hence the hyperfunction equation for the solution $Z(\lambda) = \{Z(\lambda,t)\}$ of the wave equation is

$$\text{(i)} \quad Z''(\lambda) - \alpha^2 s^2 Z(\lambda) = -\alpha^2\psi(\lambda) - \alpha^2 s\phi(\lambda).$$

It is to be remarked that the right hand side of the equation is a *linear function of* s.

The heat equation shall also be discussed in the domain $\lambda_1 \leqq \lambda \leqq \lambda_2$, $t \geqq 0$. Then, by (5.6), we obtain

$$sZ(\lambda) = Z_t(\lambda,t) + Z(\lambda,0),$$

and hence, putting $Z(\lambda,0) = \phi(\lambda)$, the hyperfunction equation for the solution $Z(\lambda) = \{Z(\lambda,t)\}$ of the heat equation is

(ii) $Z''(\lambda) - \alpha^2 sZ(\lambda) = -\alpha^2\phi(\lambda)$.

We are thus led to discuss the following problem. Let $w \in C/C$ be a *logarithmic hyperfunction*. Consider a hyperfunction-valued function $f(\lambda) = \{f(\lambda,t)\}$ which satisfies

(iii) $f''(\lambda) - wf(\lambda) = g(\lambda) \qquad (\lambda_1 \leqq \lambda \leqq \lambda_2,\ 0 \leqq t)$,

where $g(\lambda)$ is a linear function of w, the coefficients of this linear function being numerical valued functions of λ.

The case $w = s$ is the wave equation (Chapter IX) and the case $w = s^{1/2}$ is the heat equation (XI). We shall discuss the telegraph equation in Chapter X.

REMARK. In the case of *the vibration of a string*, let $Z(\lambda,t)$ be the value of the vertical displacement of the point of the string (with the abscissa λ, at the instant t) from the equilibrium position of the string (= the λ axis). Then the customary assumption that the partial derivative

$$\frac{\partial^2}{\partial\lambda^2} Z(\lambda,t)$$

is continuous at every point λ of the string might be somewhat restrictive from the physical point of view. In this sense, our approach, using the generalized derivative $Z''(\lambda)$, would be easier to handle as the reader will see in due course in this Part IV (see, e.g., Remark 30.1 in the Section 30 of this book).