Part II
Convergence Theory

The following abstract setting underpins the special problems we have considered up to now. To begin with, suppose a given problem can be formulated as an operator equation

$$Au = w,$$

with A, a mapping between suitable spaces E and F. Then, suppose we can construct approximate solutions, which again are expressed as solutions of operator equations

$$A_n u_n = w_n, \quad n \in \mathbb{N},$$

where A_n, $n \in \mathbb{N}$, represents a sequence of mappings between spaces E_n and F_n.

The concrete settings arising in the problems considered in Part I of this book can be depicted graphically by the following typical figure,

$$\begin{array}{ccc} E & \xrightarrow{\quad A \quad} & F \\ {\scriptstyle R_n^E}\big\downarrow & & \big\downarrow{\scriptstyle R_n^F} \\ E_n & \xrightarrow{\quad A_n \quad} & F_n \end{array}$$

E_n and F_n are either subspaces of E and F, respectively, e.g., in case of projection methods where R_n^E and R_n^F can be chosen as projection operators (cf. Chapter 2 and Section 3.3). Or, E_n and F_n are spaces of grid functions where, obviously, no subspace situation is present, and

R_n^E and R_n^F are essentially pointwise restrictions (cf. Chapter 1 and Section 3.2). Moreover, a more complicated situation arises in case of initial value problems where we have termed the spaces X, X_n, Y, Y_n and the corresponding R_n^X, R_n^Y will be specified in Chapter 11.

The solvability of the given problem and that of the approximate problems have already been discussed for the most part. Now, three fundamental questions arise:

(i) How do the spaces E_n and F_n, $n \in \mathbb{N}$, approximate the spaces E and F, respectively?

(ii) In what sense does the sequence A_n, $n \in \mathbb{N}$, represent an approximation to A?

And, most importantly,

(iii) In what sense are the elements of the sequence u_n, $n \in \mathbb{N}$, approximations to the solution u of the given problem?

To address these questions, we first define a concept of convergence, which is meaningful for all the problems and approximation methods that we have considered up to now. This aim can be achieved by the concept of "discrete convergence", which is introduced in Chapter 5 and illustrated by means of a series of examples. In Chapter 6, we develop the salient theorems on the discrete convergence of mappings and solutions to linear and nonlinear operator equations. In Chapter 7, we prove sufficient and in some cases - necessary convergence criteria formulated in terms of compactness properties associated with the sequence of operators A_n, $n \in \mathbb{N}$.

In order to make the study of the material presented in this part of the book easier, we want to point out that only Section 5.1 from Chapter 5 is needed for the stability and convergence theory developed in Chapters 6 and 7. The other sections in Chapter 5 treat examples of discrete approximations - and verify the corresponding properties - which, however, present the underlying framework for the problem areas analyzed in Parts III and IV of this book. For a first understanding of the theory of discrete approximations and discrete convergence, we therefore suggest to study Section 5.1 and the rather general, but trivial examples in Section 5.2 and skip Sections 5.3 and 5.4 for the time being.