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V.I. Arnold

Geometrical Methods in the Theory of Ordinary Differential Equations

Second Edition

Translated by Joseph Szücs
English Translation Edited by Mark Levi

With 162 Illustrations

Springer

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Preface to the Second Edition

Since 1978, when the first Russian edition of this book appeared, geometrical methods in the theory of ordinary differential equations have become very popular. A lot of computer experiments have been performed and some theorems have been proved. In this edition, this progress is (partially) represented by some additions to the first English text. I mention here some of these recent discoveries.

1. The Feigenbaum universality of period doubling cascades and its extensions—the renormalization group analysis of bifurcations (Percival, Landford, Sinai, ...).
2. The Żołądek solution of the two-parameter bifurcation problem (cases of two imaginary pairs of eigenvalues and of a zero eigenvalue and a pair).
3. The Iljashenko proof of the "Dulac theorem" on the finiteness of the number of limit cycles of polynomial planar vector fields.
4. The Ecalle and Voronin theory of holomorphic invariants for formally equivalent dynamical systems at resonances.
5. The Varchenko and Hovanski theorems on the finiteness of the number of limit cycles generated by a polynomial perturbation of a polynomial Hamiltonian system (the Dulac form of the weakened version of Hilbert's sixteenth problem).
6. The Petrov estimates of the number of zeros of the elliptic integrals responsible for the birth of limit cycles for polynomial perturbations of the Hamiltonian system $\ddot{x} = x^2 - 1$ (solution of the weakened sixteenth Hilbert problem for cubic Hamiltonians).
7. The Bachtin theorems on averaging in systems with several frequencies.
8. The Davydov theory of normal forms for singularities of implicit differential equations and relaxation oscillations.
9. The Neistadt and Cary–Escande–Tennyson theory of adiabatic invariant's change under separatrix crossings (explaining, according to Wisdom, the Kirkwood gaps in the distribution of asteroids).
10. The Neistadt theory of dynamical bifurcations.

The problem of bifurcations at 1:4 resonance seems to be still unsolved, but I present the conjectural answer supported by both computer experiments and asymptotic analysis.

I mention here some other important recent results:

(1) the bifurcation theory of fundamental systems of solutions of linear equations (related to the Schubert stratification of the Grassmannians and to the Weierstrass points on algebraic curves) by M. Kazarian;
(2) the theory of normal forms of vector fields with Jordan linear part (related to sl(2)-modules) by Bogaevski, Povzner, and Givental;
(3) the bifurcation theory of cycles in reversible systems (related to the metamorphoses of the umbrella's sections) by M. Sevrjuk;
(4) the theory of nonoscillatory linear equations (related to the geometry of the Schubert stratification of the flag manifolds);
(5) the classification of the local topological bifurcations in generic gradient systems depending on three parameters (related to the Thom conjecture on catastrophes) by B. Hessin;
(6) the theory of versal deformations for the vector fields on a line (related to the differential forms of complex degree) by V. Kostov;
(7) the tunelling asymptotics in systems with many competing attractors (related to the Fokker–Planck equation and to the Witten inequalities) by V. Fok;
(8) the bifurcation theory for planar homogeneous vector fields (related to the higher dimensional umbrellas) by B. Hessin.

These subjects need too much space to be discussed here.

The reader may find more details and an extensive bibliography, for many of the subjects discussed in this textbook, in *The Encyclopaedia of Mathematical Sciences*, Volumes 1, 3, and 5, also published by Springer-Verlag.

Moscow
September 10, 1987

V. Arnold

Preface to the First Edition

Newton's fundamental discovery, the one which he considered necessary to keep secret and published only in the form of an anagram, consists of the following: *Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa.* In contemporary mathematical language, this means: "It is useful to solve differential equations".

At present, the theory of differential equations represents a vast conglomerate of a great many ideas and methods of different nature, very useful for many applications and constantly stimulating theoretical investigations in all areas of mathematics.

Many of the routes connecting abstract mathematical theories to applications in the natural sciences lead through differential equations. Many topics of the theory of differential equations grew so much that they became disciplines in themselves; problems from the theory of differential equations had great significance in the origins of such disciplines as linear algebra, the theory of Lie groups, functional analysis, quantum mechanics, etc. Consequently, differential equations lie at the basis of scientific mathematical philosophy (Weltanschauung).

In the selection of material for this book, the author intended to expound basic ideas and methods applicable to the study of differential equations. Special efforts were made to keep the basic ideas (which are, as a rule, simple and intuitive) free from technical details. The most fundamental and simple questions are considered in the greatest detail, whereas the exposition of the more special and difficult parts of the theory has been given the character of a survey.

The book begins with the study of some special differential equations integrable by quadrature. Attention is paid mainly to connections with general mathematical ideas, methods, and concepts (resolution of singularities, Lie groups, and Newton diagrams) on the one hand, and to applications to the natural sciences on the other, rather than to the formal cookbook aspect of the elementary theory of integration.

The theory of partial differential equations of the first order is considered by means of the natural contact structure in the manifold of 1-jets of functions. The necessary elements of the geometry of contact structures are developed incidentally, making the entire theory independent of other sources.

A significant portion of the book is concerned with methods which are usually called *qualitative*. The recent development of the qualitative theory of differential equations, originated by Poincaré, led to the realization that similar to the fact that the explicit integration of differential equations is generally impossible, the qualitative study of general differential equations with a multidimensional phase space turns out to be impossible. The book discusses the analysis of differential equations from the point of view of structural stability, that is, the stability of the qualitative picture with respect to a small change in the differential equations. The basic results obtained after the first publications of Andronov and Pontrjagin in this area are expounded: the elements of the theory of structurally stable Anosov systems, all trajectories of which are exponentially unstable, and Smale's theorem on the nondensity of structurally stable systems. We also discuss the significance of these mathematical discoveries to applications. (We speak of the description of stable chaotic regimes of motion like turbulence.)

The most powerful and frequently applicable methods of study of differential equations are the various asymptotic methods. We develop the basic ideas of the averaging method going back to the work of the founders of celestial mechanics and widely usable in all those areas of application, where a slow evolution has to be separated from fast oscillations (Bogoljubov, Mitropol'skiĭ, and others).

In spite of the abundant research in averaging, in the problem of evolution even for the simplest multifrequency systems, everything is not entirely clear. We give a survey of the work concerning passage through resonances and capture to resonance in an attempt to illuminate the problem.

The basis of the averaging method is the idea of annihilating perturbations by means of an appropriate choice of the coordinate system. This very idea lies at the basis of the theory of Poincaré normal forms. The method of normal forms is the fundamental method of the local theory of differential equations, which describes the behavior of phase curves in the neighborhood of a singular point or a closed phase curve. In this book, we describe the main results of the method of Poincaré normal forms, including a proof of Siegel's fundamental theorem on the linearization of a holomorphic mapping.

Important applications of the method of Poincaré normal forms come across not only in the study of a single differential equation, but also in bifurcation theory, where the subject of research is a family of equations depending on parameters.

Bifurcation theory studies the qualitative change under the variation of the parameters on which the system depends. For general values of the parameters, we usually have to deal with generic systems (all singular points are simple, etc.). However, if a system depends on parameters, then for some values of the parameters we cannot avoid degeneracies (for example, the fusion of two singular points of a vector field).

In a one-parameter system, we generically encounter only simple degeneracies (those which we cannot get rid of by a small perturbation of the family). Consequently, there arises a hierarchy of degeneracies according to the codimensions of the corresponding surfaces in the function space of all systems under study: in one-parameter generic families, only degeneracies corresponding to surfaces of codimension 1 occur, and so on.

Recent progress in bifurcation theory is connected with the application of ideas and methods of the general theory of singularities of differentiable mappings due to Whitney.

This book concludes with a chapter on bifurcation theory, in which the methods developed in the preceding chapters are applied, and main results obtained in this field, beginning with the fundamental work of Poincaré and Andronov, are described.

In discussing all of these subjects, the author attempts to avoid the axiomatic-deductive style, with its unmotivated definitions concealing the fundamental ideas and methods; similar to parables, they are explained only to disciples in private.

The axiomization and algebraization of mathematics, after more than 50 years, has led to the illegibility of such a large number of mathematical texts that the threat of complete loss of contact with physics and the natural sciences has been realized. The author attempts to write in such a way that this book can be read by not only mathematicians, but also all users of the theory of differential equations.

We only assume a little general mathematical knowledge on the part of the reader, let us say roughly the first two courses of a university program; for example, familiarity with the textbook V. I. Arnold, *Ordinary Differential Equations*, Moscow, Nauka, 1974 [in English, Cambridge, MA, MIT Press, 1973, 1978]* is sufficient (but not necessary).

The exposition is developed in such a way that the reader can omit passages that turn out to be difficult for him, without much harm to the understanding of what follows: as much as possible, we avoid references from one chapter to another, and even from one paragraph to another.

The content of this book constitutes the material of a series of mandatory and special courses delivered by the author at the Department of Mechanics and Mathematics of Moscow State University, 1970–1976, to students of mathematics in grades II–III, and to mathematicians working in applications.

The author expresses his gratitude to students O. E. Hadin, A. K. Koval'dzhi, E. M. Kaganova, and to Professor Ju. S. Il'jašenko, whose notes were very useful in the preparation of this book. The notes of a special course composed by Il'jašenko and the notes of the lectures given in the experimental group have been in the department library for a number of

* In the exposition of some special questions, we have also used or recalled elementary information on differential forms, Lie groups, and functions of a complex variable. This information is not necessary for the understanding of most of the book.

years. The author is grateful to the many readers and students of these courses for a series of valuable remarks used in the preparation of the book. The author is grateful to referees D. V. Anosov and V. A. Pliss for a careful and helpful review of the manuscript.

June, 1977 V. ARNOLD

Contents

Notation

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}^n$	the n-dimensional real linear space
$\exists$	there exists
$\forall$	for every
$a \in A$	the element a of the set A
$A \subset B$	the subset A of the set B
$A \cap B$	intersection of the sets A and B
$A \cup B$	union of the sets A and B
$A \backslash B$	difference of the sets A and B (the part of A outside B)
$A \times B$	direct product of the sets A and B (the set of pairs (a, b), $a \in A, b \in B$)
$A \oplus B$	direct sum of linear spaces
$f: A \to B$	a mapping f of A into B
$x \mapsto y$ or $y = f(x)$	the mapping f maps the element x onto the element y
$\operatorname{Im} f$ or $f(A)$	image under the mapping f (but $\operatorname{Im} z$ is the imaginary part of z)
$f^{-1}(y)$	complete inverse image of the point y under the mapping f (the set of all x for which $f(x) = y$)
$\operatorname{Ker} f$	kernel of the linear operator f (the complete inverse image of zero)
$\dot{f}$	rate of change of the function f (derivative with respect to time t)
$f', f_*, df/dx, Df/Dx$	derivative of the mapping f
$T_x M$	the tangent space of the manifold M at the point x
$A \Rightarrow B$	assertion A implies B
$A \Leftrightarrow B$	assertions A and B are equivalent
$\omega_1 \wedge \omega_2$	exterior product of the differential forms ω_1 and ω_2
$f \circ g$	composition of mappings $[(f \circ g)(x) = f(g(x))]$
$L_v f$	derivative of the function f in the direction of the vector field v