Graduate Texts in Mathematics 117

Springer Science+Business Media, LLC

Graduate Texts in Mathematics

1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
2 OXTOBY. Measure and Category. 2nd ed.
3 SCHAEFER. Topological Vector Spaces.
4 HILTON/STAMMBACH. A Course in Homological Algebra. 2nd ed.
5 MAC LANE. Categories for the Working Mathematician.
6 HUGHES/PIPER. Projective Planes.
7 SERRE. A Course in Arithmetic.
8 TAKEUTI/ZARING. Axiomatic Set Theory.
9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
10 COHEN. A Course in Simple Homotopy Theory.
11 CONWAY. Functions of One Complex Variable I. 2nd ed.
12 BEALS. Advanced Mathematical Analysis.
13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
16 WINTER. The Structure of Fields.
17 ROSENBLATT. Random Processes. 2nd ed.
18 HALMOS. Measure Theory.
19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
20 HUSEMOLLER. Fibre Bundles. 3rd ed.
21 HUMPHREYS. Linear Algebraic Groups.
22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
23 GREUB. Linear Algebra. 4th ed.
24 HOLMES. Geometric Functional Analysis and Its Applications.
25 HEWITT/STROMBERG. Real and Abstract Analysis.
26 MANES. Algebraic Theories.
27 KELLEY. General Topology.
28 ZARISKI/SAMUEL. Commutative Algebra. Vol.I.
29 ZARISKI/SAMUEL. Commutative Algebra. Vol.II.
30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
33 HIRSCH. Differential Topology.
34 SPITZER. Principles of Random Walk. 2nd ed.
35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
37 MONK. Mathematical Logic.
38 GRAUERT/FRITZSCHE. Several Complex Variables.
39 ARVESON. An Invitation to C^*-Algebras.
40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
42 SERRE. Linear Representations of Finite Groups.
43 GILLMAN/JERISON. Rings of Continuous Functions.
44 KENDIG. Elementary Algebraic Geometry.
45 LOÈVE. Probability Theory I. 4th ed.
46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimensions 2 and 3.
48 SACHS/WU. General Relativity for Mathematicians.
49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
50 EDWARDS. Fermat's Last Theorem.
51 KLINGENBERG. A Course in Differential Geometry.
52 HARTSHORNE. Algebraic Geometry.
53 MANIN. A Course in Mathematical Logic.
54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
56 MASSEY. Algebraic Topology: An Introduction.
57 CROWELL/FOX. Introduction to Knot Theory.
58 KOBLITZ. p-adic Numbers, p-adic Analysis, and Zeta-Functions. 2nd ed.
59 LANG. Cyclotomic Fields.
60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.

continued after index

Jean-Pierre Serre

Algebraic Groups and Class Fields

Translation of the French Edition

Springer

Jean-Pierre Serre
Professor of Algebra and Geometry
Collège de France
75231 Paris Cedex 05
France

AMS Classifications: 11G45 11R37

LCCN 87-32121

This book is a translation of the French edition: *Groupes algébriques et corps de classes*. Paris: Hermann, 1975.

Originally published by Springer-Verlag New York Inc. in 1988
Softcover reprint of the hardcover 1st edition 1988

Text prepared in camera-ready form using TEX.

9 8 7 6 5 4 3 2 (Corrected second printing, 1997)

ISBN 978-1-4612-6993-9 ISBN 978-1-4612-1035-1 (eBook)
DOI 10.1007/978-1-4612-1035-1

Contents

CHAPTER I
Summary of Main Results 1

1. Generalized Jacobians 1
2. Abelian coverings 3
3. Other results 4

Bibliographic note 5

CHAPTER II
Algebraic Curves 6

1. Algebraic curves 6
2. Local rings 7
3. Divisors, linear equivalence, linear series 8
4. The Riemann-Roch theorem (first form) 10
5. Classes of répartitions 11
6. Dual of the space of classes of répartitions 12
7. Differentials, residues 14
8. Duality theorem 16
9. The Riemann-Roch theorem (definitive form) 17
10. Remarks on the duality theorem 18
11. Proof of the invariance of the residue 19
12. Proof of the residue formula 21
13. Proof of lemma 5 23

Bibliographic note 25

CHAPTER III
Maps From a Curve to a Commutative Group 27

§1. Local symbols 27
1. Definitions 27
2. First properties of local symbols 30
3. Example of a local symbol: additive group case 33
4. Example of a local symbol: multiplicative group case 34

§2. Proof of theorem 1 37
5. First reduction 37
6. Proof in characteristic 0 38
7. Proof in characteristic $p > 0$: reduction of the problem 40
8. Proof in characteristic $p > 0$: case a) 41
9. Proof in characteristic $p > 0$: reduction of case b) to the unipotent case 42
10. End of the proof: case where G is a unipotent group 43

§3. Auxiliary results 45
11. Invariant differential forms on an algebraic group 45
12. Quotient of a variety by a finite group of automorphisms 48
13. Some formulas related to coverings 51
14. Symmetric products 53
15. Symmetric products and coverings 54
Bibliographic note 56

CHAPTER IV
Singular Algebraic Curves 58

§1. Structure of a singular curve 58
1. Normalization of an algebraic variety 58
2. Case of an algebraic curve 59
3. Construction of a singular curve from its normalization 60
4. Singular curve defined by a modulus 61

§2. Riemann-Roch theorems 62
5. Notations 62
6. The Riemann-Roch theorem (first form) 63
7. Application to the computation of the genus of an algebraic curve 64
8. Genus of a curve on a surface 65

§3. Differentials on a singular curve 68
9. Regular differentials on X' 68
10. Duality theorem 70
11. The equality $n_Q = 2\delta_Q$ 71
12. Complements 72
Bibliographic note 73

CHAPTER V

Generalized Jacobians

Generalized Jacobians 74

§1. Construction of generalized Jacobians 74
1. Divisors rational over a field 74
2. Equivalence relation defined by a modulus 76
3. Preliminary lemmas 77
4. Composition law on the symmetric product $X^{(\pi)}$ 79
5. Passage from a birational group to an algebraic group 80
6. Construction of the Jacobian $J_{\mathfrak{m}}$ 81

§2. Universal character of generalized Jacobians 82
7. A homomorphism from the group of divisors of X to $J_{\mathfrak{m}}$ 82
8. The canonical map from X to $J_{\mathfrak{m}}$ 84
9. The universal property of the Jacobians $J_{\mathfrak{m}}$ 87
10. Invariant differential forms on $J_{\mathfrak{m}}$ 89

§3. Structure of the Jacobians $J_{\mathfrak{m}}$ 90
11. The usual Jacobian 90
12. Relations between Jacobians $J_{\mathfrak{m}}$ 91
13. Relation between $J_{\mathfrak{m}}$ and J 91
14. Algebraic structure on the local groups $U/U^{(n)}$ 92
15. Structure of the group $V_{(n)}$ in characteristic zero 94
16. Structure of the group $V_{(n)}$ in characteristic $p > 0$ 94
17. Relation between $J_{\mathfrak{m}}$ and J: determination of the algebraic structure of the group $L_{\mathfrak{m}}$ 96
18. Local symbols 98
19. Complex case 99

§4. Construction of generalized Jacobians: case of an arbitrary base field 102
20. Descent of the base field 102
21. Principal homogeneous spaces 104
22. Construction of the Jacobian $J_{\mathfrak{m}}$ over a perfect field 105
23. Case of an arbitrary base field 107
Bibliographic note 108

CHAPTER VI

Class Field Theory

Class Field Theory 109

§1. The isogeny $x \to x^q - x$ 109
1. Algebraic varieties defined over a finite field 109
2. Extension and descent of the base field 110
3. Tori over a finite field 111
5. Quadratic forms over a finite field 114
6. The isogeny $x \to x^q - x$: commutative case 115

§2. Coverings and isogenies 117
7. Review of definitions about isogenies 117
8. Construction of coverings as pull-backs of isogenies 118
9. Special cases 119
10. Case of an unramified covering 120
11. Case of curves 121
12. Case of curves: conductor 122

§3. Projective system attached to a variety 124
13. Maximal maps 124
14. Some properties of maximal maps 127
15. Maximal maps defined over k 129

§4. Class field theory 130
16. Statement of the theorem 130
17. Construction of the extensions E_α 132
18. End of the proof of theorem 1: first method 134
19. End of the proof of theorem 1: second method 135
20. Absolute class fields 137
21. Complement: the trace map 138

§5. The reciprocity map 139
22. The Frobenius substitution 139
23. Geometric interpretation of the Frobenius substitution 140
24. Determination of the Frobenius substitution in an extension of type α 141
25. The reciprocity map: statement of results 142
26. Proof of theorems 3, 3′, and 3″ starting from the case of curves 144
27. Kernel of the reciprocity map 145

§6. Case of curves 146
28. Comparison of the divisor class group and generalized Jacobians 146
29. The idèle class group 149
30. Explicit reciprocity laws 150

§7. Cohomology 152
31. A criterion for class formations 152
32. Some properties of the cohomology class $u_{F/E}$ 155
33. Proof of theorem 5 156
34. Map to the cycle class group 157

Bibliographic note 159

CHAPTER VII

Group Extension and Cohomology 161

§1. Extensions of groups 161

1. The groups $\mathrm{Ext}(A, B)$ 161
2. The first exact sequence of Ext 164
3. Other exact sequences 165
4. Factor systems 166
5. The principal fiber space defined by an extension 168
6. The case of linear groups 169

§2. Structure of (commutative) connected unipotent groups 171
7. The group $\mathrm{Ext}(\mathbf{G}_a, \mathbf{G}_a)$ 171
8. Witt groups 171
9. Lemmas 173
10. Isogenies with a product of Witt groups 175
11. Structure of connected unipotent groups: particular cases 177
12. Other results 178
13. Comparison with generalized Jacobians 179

§3. Extensions of Abelian varieties 180
14. Primitive cohomology classes 180
15. Comparison between $\mathrm{Ext}(A, B)$ and $H^1(A, \mathcal{B}_A)$ 181
16. The case $B = \mathbf{G}_m$ 183
17. The case $B = \mathbf{G}_a$ 184
18. Case where B is unipotent 186

§4. Cohomology of Abelian varieties 187
19. Cohomology of Jacobians 187
20. Polar part of the maps $\varphi_{\mathfrak{m}}$ 190
21. Cohomology of Abelian varieties 190
22. Absence of homological torsion on Abelian varieties 192
23. Application to the functor $\mathrm{Ext}(A, B)$ 195
Bibliographic note 196

Bibliography 198

Supplementary Bibliography 204

Index 206