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Generalized Solutions of First-Order PDEs

The Dynamical Optimization Perspective

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Introduction

Hamilton–Jacobi equations and other types of partial differential equations of the first order are dealt with in many branches of mathematics, mechanics, and physics. These equations are usually nonlinear, and functions vital for the considered problems are not smooth enough to satisfy these equations in the classical sense.

An example of such a situation can be provided by the value function of a differential game or an optimal control problem. It is known that at the points of differentiability this function satisfies the corresponding Hamilton–Jacobi–Isaacs–Bellman equation. On the other hand, it is well known that the value function is as a rule not everywhere differentiable and therefore is not a classical global solution. Thus in this case, as in many others where first-order PDE's are used, there arises necessity to introduce a notion of generalized solution and to develop theory and methods for constructing these solutions.

In the 50s–70s, problems that involve nonsmooth solutions of first-order PDE's were considered by Bakhvalov, Evans, Fleming, Gel'fand, Godunov, Hopf, Kuznetzov, Ladyzhenskaya, Lax, Oleinik, Rozhdestvenskiĭ, Samarskii, Tikhonov, and other mathematicians. Among the investigations of this period we should mention the results of S.N. Kruzhkov, which were obtained for Hamilton–Jacobi equation with convex Hamiltonian. A review of the investigations of this period is beyond the limits of the present book. A sufficiently complete bibliography can be found in [58, 126, 128, 141].

In the early 80s, M.G. Crandall and P.-L. Lions introduced the concept of viscosity solution. The first publications [54, 57, 58] were followed by an extensive series of papers of many authors. The theory of viscosity solutions has helped advance the investigations of first-order PDE's and elliptic equations. Within this theory uniqueness and existence theorems have been developed for various types of equations and boundary-value problems, and also some applications to control problems and differential games have been studied. A review of the results of the theory of viscosity solutions is given in [56]. In the present book we shall give the definition of a viscosity solution and also some facts of the theory of viscosity solutions.

In this book we develop another approach that can be considered as a nonclassical method of characteristics, according to which the generalized solution (called the minimax one) is assumed to be flow invariant with respect to the so-called characteristic inclusions. The term "minimax solution" originates from the theory of differential games. In the early 70s, N.N. Krasovskiĭ and the author of the present book introduced u-stable and v-stable functions, which majorize and minorize the value function (cf., e.g., [118, 120, 122]). The value function of a differential game is the only function which is simultaneously u- and v-stable. It is also known that at the points of differentiability the value function satisfies a first-order PDE (Isaacs-Bellman equation). Thus the mentioned properties determine one and only one generalized (minimax) solution of the Isaacs-Bellman equation. Let us note that u-stable [v-stable] functions coincide with viscosity supersolutions [subsolutions] of the Isaacs-Bellman equation. The properties of u- and v-stability can be formalized in different ways and, in particular, with the help of inequalities for directional derivatives. These inequalities were introduced in the articles [192, 203], which were published in 1978 and 1980 and, most probably, were the first to define a generalized solution of first-order PDE by substituting the equation by a pair of differential inequalities.

The use of the term "minimax solution" is justified by permanent presence of minimax operations in investigations of these solutions, including the well-known Hopf formulas [90], and in investigations based on idempotent analysis, which have been implemented in recent years by V.P. Maslov, V.N. Kolokol'tsov, and S.N. Samborskiĭ. The conception of generalized solution proposed in works of this direction (cf., e.g., [102, 147]) is similar to the classical approaches to defining generalized (weak) solutions in mathematical physics. The main difference is that the traditional field structure in $\mathbf{R}$ with operations $a + b$ and $a \cdot b$ is substituted by the structure of semiring with operations $a \oplus b = \min(a, b)$, $a \odot b = a + b$. According to this approach the "scalar product" of functions f and g is defined by the equality $(f, g) = \inf_{x \in X}(f(x) + g(x))$. Based on this approach, investigations of first-order PDE's with convex Hamiltonian have been developed, and applications to problems of mathematical physics have been obtained.

The book is intended as a self-contained exposition of the theory of minimax solutions. It includes existence and uniqueness results, examples of numerical modelling, applications to the theory of control and differential games. The research on minimax solutions employs methods of nonsmooth analysis, Lyapunov functions, dynamical optimization, and the theory of differential games; at the same time, this research can contribute to the development of these branches of mathematics. At an infinitesimal level one can notice duality of different approaches to defining generalized solutions. This confirms the remark of L. Young [227]: "... Actually, Hamiltonians are inseparably intertwined with the notion of convexity and particularly with duality of convex figures".

The book consists of 4 chapters and contains 20 sections and an appendix. Each section begins with an abstract. The structure of the book can be seen from the table of contents, and we will not retell the book in this introduction.

Let us describe briefly the way in which we introduce the notion of minimax solution of the equation

$$F(x, u(x), Du(x)) = 0, \quad x \in G \subset \mathbf{R}^n, \tag{1}$$

where G is an open set in $\mathbf{R}^n$, Du is the gradient of real function u. The classical method of characteristics prompts introducing the ordinary differential equation

$$\dot{z}(t) = \langle \dot{x}(t), s \rangle - F(x(t), z(t), s). \tag{2}$$

A minimax solution of PDE (1) is understood as a continuous function $u : G \mapsto \mathbf{R}$ whose graph is weakly invariant (synonyms: flow invariant or viable) with respect to equation (2), that is, for every $(x_0, z_0) \in \operatorname{gr} u := \{(x, u(x)) : x \in G\}$ and $s \in \mathbf{R}^n$ there exist a number $\tau > 0$ and a Lipschitz function $(x(\cdot), z(\cdot)) : [0, \tau] \mapsto G \times \mathbf{R}$ such that $(x(0), z(0)) = (x_0, z_0)$, $z(t) = u(x(t))$ and that equation (2) is satisfied for almost all $t \in [0, \tau]$.

This definition is somewhat akin to the ideas of the above mentioned duality, the Legendre transformation, and some constructions of dynamical optimization introduced in publications by Clarke, Fleming, Krasovskiĭ, Krotov, and Rockafellar.

Of the results on differential games considered in the present book let us mention the construction of suboptimal strategies, which is similar to the well-known definition of optimal strategy in the framework of the classical method of dynamical programming in the case when the value function of differential game is smooth. The difference is that the gradient of value function (which may fail to exist) is substituted by the quasi-gradient. The definition of the quasi-gradient is based on results of the theory of generalized solutions of first-order PDE's. This result is an example of application of the theory of generalized solutions and nonsmooth analysis to the theory of differential games.

An appendix at the end of the book gives the necessary facts, mostly of nonsmooth analysis and the theory of differential inclusions. Let us mention, in particular, a criterion for weak invariance (viability), which in contrast to the well-known results, instead of the Bouligand cone uses its convex hull.

In the appendix we also formulate and prove a property of subdifferential, which is essential for investigating minimax solutions. Let us mention that this property can be considered as one more example showing reciprocal influence of investigations in the theory of differential games, theory of first-order PDE's, and nonsmooth analysis. A result close to this property was obtained for the first time in article [84], which dealt with differential games (Lemma 4.3 proven by V.N. Ushakov). Then this result was further developed in [205, 196] in connection with investigation of minimax solutions. Further, the article [196] was the original stimulus for an important result of nonsmooth analysis — mean-value inequality obtained recently by F. Clarke and Yu.S. Ledyaev [49, 48]. This result is a "multidirectional" generalization in nonsmooth setting of the classical mean value theorem.

Brief bibliographical comments are given in the book. They deal with the works most closely connected with the results presented in monograph. We do not claim that these comments are complete.

The present book can be considered as a sequel and development of the author's monograph "Minimax Inequalities and Hamilton–Jacobi Equations", Nauka, Moscow, 1991 (in Russian). The present book contains also some results of the monograph by N.N. Krasovskiĭ and the author "Game-Theoretical Control Problems", Springer, New York, 1988.

I wish to express deep gratitude to my teacher N.N. Krasovskiĭ. His attention and valuable advice are essential for my investigations and have been very helpful in the work on the book.

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NOTATION

The following notations are assumed:

$\mathbf{R}^n$ is the n-dimensional Euclidean space with elements $x = (x_1, \dots, x_n)$;

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

is the inner product of $x, y \in \mathbf{R}^n$; $\|x\| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of $x \in \mathbf{R}^n$.

The closure, boundary, and convex hull of a set $G \subset \mathbf{R}^n$ are denoted by $\overline{G}$, ∂G, and $\operatorname{co} G$, respectively;

$$B(y; r) = \{x \in \mathbf{R}^n : \|x - y\| \leq r\}$$

is the Euclidean ball of radius r with center y; the symbol B_r denotes the ball $\{x \in \mathbf{R}^n : \|x\| \leq r\}$. We denote intervals of $\mathbf{R}^1$ respectively open, closed, and half-open by (a, b), $[a, b]$, $(a, b]$, $[a, b)$. We also use the notation $\mathbf{R}^+ = [0, \infty)$.

For a real function $\mathbf{R}^n \ni x \mapsto h(x) \in \mathbf{R}$ and a set $X \subset \mathbf{R}^n$ we let $\operatorname{Arg\,min}_{x \in X} h(x)$ and $\operatorname{Arg\,max}_{x \in X} h(x)$ denote the sets of minimizing and maximizing elements, i.e.,

$$\operatorname{Arg\,min}_{x \in X} h(x) = \{x_0 \in X : h(x_0) \leq h(x) \ \forall x \in X\},$$
$$\operatorname{Arg\,max}_{x \in X} h(x) = \{x_0 \in X : h(x_0) \geq h(x) \ \forall x \in X\}.$$

The symbols $\sup h(X)$ and $\inf h(X)$ stand for

$$\sup h(X) = \sup_{x \in X} h(x), \quad \inf h(X) = \inf_{x \in X} h(x).$$

The symbol $\operatorname{gr} u$ (respectively, $\operatorname{epi} u$ and $\operatorname{hypo} u$) denotes the graph (respectively, epigraph and hypograph) of a function $u : G \mapsto \mathbf{R}$. Namely,

$$\begin{aligned} \operatorname{gr} u &= \{(x, z) : z = u(x), \ x \in G\}, \\ \operatorname{epi} u &= \{(x, z) : z \geq u(x), \ x \in G\}, \\ \operatorname{hypo} u &= \{(x, z) : z \leq u(x), \ x \in G\}. \end{aligned}$$

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