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(continued after index)

John L. Troutman

Variational Calculus and Optimal Control

Optimization with Elementary Convexity

Second Edition

With 87 Illustrations

Springer

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This book is dedicated to my parents
and to my teachers

Preface

Although the calculus of variations has ancient origins in questions of Aristotle and Zenodoros, its mathematical principles first emerged in the post-calculus investigations of Newton, the Bernoullis, Euler, and Lagrange. Its results now supply fundamental tools of exploration to both mathematicians and those in the applied sciences. (Indeed, the macroscopic statements obtained through variational principles may provide the only valid mathematical formulations of many physical laws.) Because of its classical origins, variational calculus retains the spirit of natural philosophy common to most mathematical investigations prior to this century. The original applications, including the Bernoulli problem of finding the brachistochrone, require optimizing (maximizing or minimizing) the mass, force, time, or energy of some physical system under various constraints. The solutions to these problems satisfy related differential equations discovered by Euler and Lagrange, and the variational principles of mechanics (especially that of Hamilton from the last century) show the importance of also considering solutions that just provide stationary behavior for some measure of performance of the system. However, many recent applications do involve optimization, in particular, those concerned with problems in optimal control.

Optimal control is the rapidly expanding field developed during the last half-century to analyze optimal behavior of a constrained process that evolves in time according to prescribed laws. Its applications now embrace a variety of new disciplines, including economics and production planning.[1] In

[1] Even the perennial question of how a falling cat rights itself in midair can be cast as a control problem in geometric robotics! See *Dynamics and Control of Mechanical Systems*: The Falling Cat and Related Problems, by Michael Enos, Ed. American Mathematical Society, 1993.

this text we will view optimal control as a special form of variational calculus, although with proper interpretation, these distinctions can be reversed.

In either field, most initial work consisted of finding (necessary) conditions that characterize an optimal solution tacitly assumed to exist. These conditions were not easy to justify mathematically, and the subsequent theories that gave (sufficient) conditions guaranteeing that a candidate solution does optimize were usually substantially harder to implement. (Conditions that ensure existence of an optimizing solution were—and are—far more difficult to investigate, and they cannot be considered at the introductory level of this text. See [Ce].) Now, in any of these directions, the statements of most later theoretical results incorporate some form of convexity in the defining functions (at times in a disguised form). Of course, convexity was to be expected in view of its importance in characterizing extrema of functions in ordinary calculus, and it is natural to employ this central theme as the basis for an introductory treatment.

The present book is both a refinement and an extension of the author's earlier text, *Variational Calculus with Elementary Convexity* (Springer-Verlag, 1983) and its supplement, *Optimal Control with Elementary Convexity* (1986). It is addressed to the same audience of junior to first-year graduate students in the sciences who have some background in multidimensional calculus and differential equations. The goal remains to solve problems completely (and exactly) whenever possible at the mathematical level required to formulate them. To help achieve this, the book incorporates a sliding scale-of-difficulty that allows its user to become gradually more sophisticated, both technically and theoretically. The few starred (*) sections, examples, and problems outside this scheme can usually be overlooked or treated lightly on first reading.

For our purposes, a convex function is a differentiable real-valued function whose graph lies above its tangent planes. In application, it may be enough that a function of several variables have this behavior only in some of the variables, and such "elementary" convexity can often be inferred through pattern recognition. Moreover, with proper formulation, many more problems possess this convexity than is popularly supposed. In fact, using only standard calculus results, we can solve most of the problems that motivated development of the variational calculus, as well as many problems of interest in optimal control.

The paradigm for our treatment is as follows: Elementary convexity suggests simple sufficiency conditions that can often lead to direct solution, and they in turn inform the search for necessary conditions that hold whether or not such convexity is present. For problems that can be formulated on a fixed interval (or set) this statement remains valid even when fixed-endpoint conditions are relaxed, or certain constraints (isoperimetric or Lagrangian) are imposed. Moreover, sufficiency arguments involving elementary convexity are so natural that even multidimensional generalizations readily suggest themselves.

In Part I, we provide the standard results of variational calculus in the context of linear function spaces, together with those in Chapter 3 that use elementary convexity to establish sufficiency. In Part II, we extend this development into more sophisticated areas, including Weierstrass–Hilbert field theory of sufficiency (Chapter 9). We also give an introduction to Hamiltonian mechanics and use it in §8.8 to motivate a different means for recognizing convexity, that leads to new elementary solutions of some classical problems (including that of the brachistochrone). Throughout these parts, we derive and solve many optimization problems of physical significance including some involving optimal controls. But we postpone our discussion of control theory until Part III, where we use elementary convexity to suggest sufficiency of the Pontjragin principle before establishing its necessity in the concluding chapter.

Most of this material has been class-tested, and in particular, that of Part I has been used at Syracuse University over 15 years as the text for one semester of a year-sequence course in applied mathematics. Chapter 8 (on Hamiltonian mechanics) can be examined independently of adjacent chapters, but Chapter 7 is prerequisite to any other subsequent chapters. On the other hand, those wishing primarily an introduction to optimal control could omit both Chapters 8 and 9. The book is essentially self-contained and includes in Chapter 0 a review of optimization in Euclidean space. It does not employ the Lebesque integral, but in the Appendix we develop some necessary results about analysis in Euclidean space and families of solutions to systems of differential equations.

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I wish once more to express my appreciation to those who first made me aware of the elegance and power of variational methods—Daniel Frederick at V.P.I., M.M. Schiffer at Stanford, and C. Lanczos as author. I must also reacknowledge the contributions that William Hrusa (now at Carnegie-Mellon University) made to the earlier work during his student days. Many of the problems in Part I and II originated with Bill, and his assistance and commentary during that initial production were invaluable. It has been rewarding to hear from those including Terry Rockafellar at the University of Washington, Frank Chorlton at the University of Aston, and Morris Kline at New York University, whose satisfaction with the earlier work helped motivate this extension.

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Syracuse, New York JOHN L. TROUTMAN

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