# Universitext



Springer Science+Business Media, LLC

# Universitext

Editors (North America): S. Axler, F.W. Gehring, and K.A. Ribet

**Aksoy/Khamsi:** Nonstandard Methods in Fixed Point Theory
**Andersson:** Topics in Complex Analysis
**Aupetit:** A Primer on Spectral Theory
**Bachman/Narici/Beckenstein:** Fourier and Wavelet Analysis
**Badescu:** Algebraic Surfaces
**Balakrishnan/Ranganathan:** A Textbook of Graph Theory
**Balser:** Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations
**Bapat:** Linear Algebra and Linear Models (2nd ed.)
**Berberian:** Fundamentals of Real Analysis
**Boltyanskii/Efremovich:** Intuitive Combinatorial Topology. (Shenitzer, trans.)
**Booss/Bleecker:** Topology and Analysis
**Borkar:** Probability Theory: An Advanced Course
**Böttcher/Silbermann:** Introduction to Large Truncated Toeplitz Matrices
**Carleson/Gamelin:** Complex Dynamics
**Cecil:** Lie Sphere Geometry: With Applications to Submanifolds
**Chae:** Lebesgue Integration (2nd ed.)
**Charlap:** Bieberbach Groups and Flat Manifolds
**Chern:** Complex Manifolds Without Potential Theory
**Cohn:** A Classical Invitation to Algebraic Numbers and Class Fields
**Curtis:** Abstract Linear Algebra
**Curtis:** Matrix Groups
**Debarre:** Higher-Dimensional Algebraic Geometry
**Deitmar:** A First Course in Harmonic Analysis
**DiBenedetto:** Degenerate Parabolic Equations
**Dimca:** Singularities and Topology of Hypersurfaces
**Edwards:** A Formal Background to Mathematics I a/b
**Edwards:** A Formal Background to Mathematics II a/b
**Farenick:** Algebras of Linear Transformations
**Foulds:** Graph Theory Applications
**Friedman:** Algebraic Surfaces and Holomorphic Vector Bundles
**Fuhrmann:** A Polynomial Approach to Linear Algebra
**Gardiner:** A First Course in Group Theory
**Gårding/Tambour:** Algebra for Computer Science
**Goldblatt:** Orthogonality and Spacetime Geometry
**Gustafson/Rao:** Numerical Range: The Field of Values of Linear Operators and Matrices
**Hahn:** Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups
**Heinonen:** Lectures on Analysis on Metric Spaces
**Holmgren:** A First Course in Discrete Dynamical Systems
**Howe/Tan:** Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathbb{R})$
**Howes:** Modern Analysis and Topology
**Hsieh/Sibuya:** Basic Theory of Ordinary Differential Equations
**Humi/Miller:** Second Course in Ordinary Differential Equations
**Hurwitz/Kritikos:** Lectures on Number Theory
**Jennings:** Modern Geometry with Applications
**Jones/Morris/Pearson:** Abstract Algebra and Famous Impossibilities

*(continued after index)*

George Bachman Lawrence Narici
Edward Beckenstein

# Fourier and Wavelet Analysis

Springer

George Bachman
Professor Emeritus
of Mathematics
Polytechnic University
5 Metrotech Center
Brooklyn, NY 11201
USA

Lawrence Narici
Department of Mathematics
and Computer Science
St. John's University
Jamaica, NY 11439
USA

Edward Beckenstein
Science Division
St. John's University
Staten Island, NY 10301
USA



Mathematics Subject Classification (2000): 42Axx, 42Cxx, 41-xx

Library of Congress Cataloging-in-Publication Data
Bachman, George, 1929–
Fourier and wavelet analysis / George Bachman, Lawrence Narici, Edward Beckenstein.
p. cm. — (Universitext)
Includes bibliographical references and index.
ISBN 978-1-4612-6793-5 ISBN 978-1-4612-0505-0 (eBook)
DOI 10.1007/978-1-4612-0505-0
1. Fourier, analysis. 2. Wavelets (Mathematics) I. Beckenstein, Edward, 1940– . II. Narici, Lawrence. III. Title. IV. Series.
QA403.5.B28 2000
515'.2433—dc21 99-36217

Printed on acid-free paper.


Originally published by Springer-Verlag New York Berlin Heidelberg in 2000
Softcover reprint of the hardcover 1st edition 2000


Production managed by Michael Koy; manufacturing supervised by Jeffrey Taub.
Camera-ready copy prepared by the authors.

9 8 7 6 5 4 3 2 (Corrected second printing, 2002)

ISBN 978-1-4612-6793-5

Not long ago many thought that the mathematical world was created out of analytic functions. It was the Fourier series which disclosed a *terra incognita* in a second hemisphere.
—E. B. van Vleck, 1914

The Fast Fourier transform—the most valuable numerical algorithm of our lifetime.
—G. Strang, 1993

... wavelets are without any doubt an exciting and intuitive concept. *This concept brings with it a new way of thinking....*
—Y. Meyer, 1993

## Foreword

Fourier, the nineteenth (and not the last!) child in his family, wanted to join an artillery regiment. His commoner status prevented it and he went on to other things. Goethe's dictum that boldness has a magic all its own found life in Fourier. He was so rash at times that his work was rejected by his peers (see the introduction to Chapter 4). He never worried about convergence and said that *any* periodic function could be expressed in a Fourier series. Nevertheless he was so original that others—Cauchy and Lagrange, among them—were inspired to attempt to place his creations on a firm foundation. They both failed. The first proof that Fourier series converged pointwise was Dirichlet's in 1829 for piecewise smooth functions (Sec. 4.6). As a result of Dirichlet's work, the idea of *function* was transmogrified. No more would it apply only to the aristocratic society of polynomials, exponentials and sines and cosines; disorderly conduct now had to be tolerated. By the mid-nineteenth century, it inspired (as a trigonometric series) Weierstrass's continuous-but-not-differentiable map (Sec. 4.3). It was such a shock at the time that Weierstrass was apparently in no hurry to disseminate it widely.

In order to generalize Dirichlet's pointwise convergence theorem for piecewise smooth functions to a wilder sort, Jordan invented the concept of 'function of bounded variation;' he proved his pointwise convergence theorem of Fourier series for such functions in 1881 (Sec. 4.6). As it became necessary to deal with this wider class of functions, the conception of integral was also transmuted. At Dirichlet's urging, it went from integral-as-antiderivative to being *defined* as area under a curve. Cauchy developed the integral from this perspective for continuous functions. Riemann extended it to discontinuous functions, although not too discontinuous.

Fejér (1904) went beyond functions of bounded variation. He discovered that for many functions $f$, $f$ can be recovered by summing the arithmetic means of its Fourier series; even if the Fourier series *diverges* at a point, the series of arithmetic means converges to $(f(t^-) + f(t^+))/2$ (Sec. 4.15). What happens at $t$'s where the one-sided limits do not exist? By removing the requirement concerning the existence of $f(t^-)$ and $f(t^+)$, Lebesgue globalized Fejér's theorem; he showed that the Fourier series for any $f \in L_1[-\pi, \pi]$ converges $(C, 1)$ to $f(t)$ a.e. The desire to do this was part of the reason that Lebesgue invented his integral; the theorem mentioned above was one of the first uses he made of it (Sec. 4.18). Denjoy, with the same motivation, extended the integral even further.

Concurrently, the emerging point of view that things could be decomposed into waves and then reconstituted infused not just mathematics but all of science. It is impossible to quantify the role that this perspective played in the development of the physics of the nineteenth and twentieth centuries, but it was certainly great. Imagine physics without it.

We develop the standard features of Fourier analysis—Fourier series, Fourier transform, Fourier sine and cosine transforms. We do NOT do it in the most elegant way. Instead, we develop it for the reader who has never seen them before. We cover more recent developments such as the discrete and fast Fourier transforms and wavelets in Chapters 6 and 7. Our treatment of these topics is strictly introductory, for the novice. (*Wavelets for idiots?*) To do them properly, especially the applications, would take at least a whole book.

What do you need to read the book? Not a lot of facts *per se*, but a little sophistication. We have helped ourselves to what we needed about the Lebesgue integral and given references. It's not much and if you don't know them exactly—if you know the analogous results (when they exist) for the Riemann integral—you can still read the book. We use some things about Hilbert space, too, and we have included a short development of what is needed in the first three short chapters. You can use them as a short course in functional analysis or start in Chapter 4 on Fourier series, referring to them on an as-needed basis. The chapters are sufficiently independent so that you *could* start in Chapter 5 (Fourier transform) or 6 (discrete, fast) or 7 (wavelets) and refer back as needed. One caveat: To appreciate Chapter 7, you should read the theory of the $L_2$ Fourier transform in Chapter 5. The $L_2$ theory is really quite pretty, anyway.

**Notation**: Our notation is all standard. On some rare occasions we use **C** for complement. If we say "by Exercise 3," we mean Exercise 3 at the end of the current section; otherwise we say "Exercise 2.4-3," meaning Exercise 3 at the end of Sec. 2.4. Hints are provided for lots (not all) of the exercises. Rather than a separate index of symbols, the symbols are blended into the index.

We prepared the book using *Scientific Word* and *Scientific Workplace.* The experience has been. . .interesting. We hope that the result is fun.

# Contents