

Symmetry and Pattern in Projective Geometry

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*There are many geometries, each describing
another world: wonderlands and Utopias,
refreshingly different from the world we
live in.*

H S M Coxeter

Preface

The methods and principles of Projective Geometry have a unique elegance. Intricate and surprising structures unfold from a very few, very simple concepts. It has been among my favorite mathematical interests for a very long time. I wrote this book because I wanted to convey to others some of the fascination I feel for this subject. It is not a ‘textbook’. It is a collection of ideas that have especially appealed to me, presented in a way that emphasizes general principles and that, I hope, will make you want to explore further. I have tried to avoid abstruse terminology and esoteric notation as much as possible, so it should be accessible to anyone with a little knowledge of matrices, determinants and vectors, and perhaps a smattering of group theory.

The characteristic of geometry that distinguishes it from other kinds of mathematics such as algebra or number theory, and that in a sense serves as a definition of ‘geometry’, is that it appeals to the imagination through intuitions arising from the way we perceive the ‘real world’. This remains so even when geometrical ideas wander into realms of higher dimensions and structures that are hard—or impossible—to ‘visualize’. So long as the link to spatial intuitions remains, however tenuous, we are still in the land of Geometry. Once that link is gone, it is no longer Geometry, it is Algebra—where the patterns and structures are abstract and have a different kind of appeal. But there is no sharp boundary.

Projective Geometry grew out of the efforts of architects and painters to represent the three-dimensional world on a flat two-dimensional surface. This is not a question of the geometry of the shapes of things as they ‘really are’, but rather the geometry of how they ‘seem to be’ when we look at them. Through the efforts of many eminent mathematicians this geometry of ‘perspective’ was developed into a whole new and exciting branch of mathematics; my aim in writing this book is to convey something of the flavor of this development, which reached its highest point in the 19th century and the first half of the 20th century. Thereafter, interest declined and projective geometry (and indeed, geometry in general) came to be regarded as quaint and ‘old-fashioned’, as mathematicians became more and more interested in abstractions. Happily, the tide seems to be turning, probably as a result of the advent of efficient

and versatile computer graphics systems which are re-establishing the link between mathematics and visual perception.

The greatest geometer of the 20th century was H S M ‘Donald’ Coxeter. Throughout an exceptionally long life he produced a torrent of beautiful and novel geometrical ideas, all expressed with inimitable elegance. All mathematicians—including me—who maintained an active interest in geometry throughout its ‘un-fashionable’ period are indebted to him.

That’s all. Hope you enjoy my book.

Bangalore, India

Eric Lord

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