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Luis Barreira • Claudia Valls

Complex Analysis  
and Differential  
Equations

 Springer

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# Preface

This book is essentially two books in one. Namely, it is an introduction to two large areas of mathematics—*complex analysis* and *differential equations*—and the material is naturally divided into two parts. This includes holomorphic functions, analytic functions, ordinary differential equations, Fourier series, and partial differential equations. Moreover, half of the book consists of approximately 200 worked-out problems plus 200 exercises of variable level of difficulty. The worked-out problems fill the gap between the theory and the exercises.

To a considerable extent, the parts of complex analysis and differential equations can be read independently. In the second part, some special emphasis is given to the applications of complex analysis to differential equations. On the other hand, the material is still developed with sufficient detail in order that the book contains an ample introduction to differential equations, and not strictly related to complex analysis.

The text is tailored to any course giving a first introduction to complex analysis or to differential equations, assuming as prerequisite only a basic knowledge of linear algebra and of differential and integral calculus. But it can also be used for independent study. In particular, the book contains a large number of examples illustrating the new concepts and results. Moreover, the worked-out problems, carefully prepared for each part of the theory, make this the ideal book for independent study, allowing the student to actually see how the theory applies, before solving the exercises.

Lisbon, Portugal

Luis Barreira and Claudia Valls

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