

Springer Monographs in Mathematics

Springer-Verlag London Ltd.

E.G. Peter Rowe

Geometrical Physics in Minkowski Spacetime

With 112 Figures



Springer

Springer Monographs in Mathematics ISSN 1439-7382

British Library Cataloguing in Publication Data

Rowe, E.G. Peter

Geometrical physics in Minkowski spacetime. - (Springer monographs in mathematics)

1. Generalized spaces 2. Space and time – Mathematics

I. Title

516.374

Library of Congress Cataloging-in-Publication Data

Rowe, E.G. Peter, 1938-1998

Geometrical physics in Minkowski spacetime / E.G. Peter Rowe.

p. cm. — (Springer monographs in mathematics)

Includes bibliographical references and index.

ISBN 978-1-84996-866-9 ISBN 978-1-4471-3893-8 (eBook)

DOI 10.1007/978-1-4471-3893-8

1. Special relativity (Physics) I. Title. II. Series.

QC173.65.R68 2000

530.11—dc21

00-061905

Mathematics Subject Classification (1991): 51B20, 83A05

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms of licences issued by the Copyright Licensing Agency. Enquiries concerning reproduction outside those terms should be sent to the publishers.

© Springer-Verlag London 2001

Originally published by Springer-Verlag London Berlin Heidelberg in 2001

Softcover reprint of the hardcover 1st edition 2001

The use of registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant laws and regulations and therefore free for general use.

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

Typesetting: Camera-ready by Marcus Tindall

12/3830-543210 Printed on acid-free paper SPIN 10778825

Foreword

Dr E.G. Peter Rowe had almost completed writing this book when his life was brutally terminated in the Yemen in December 1998.

Peter came to Durham from Canada via London in 1964 and quickly became a very popular and engagingly eccentric member of our Department. He was friendly and generous, informal, charitable, warm and full of diverse interests which, apart from mathematics and physics, ranged from anthropology and archaeology to foreign cultures and travel. This led him to spend his sabbatical leave in places such as northern Nigeria. His interest in other people's cultures took him to many parts of the world (from Ladakh in North West India, to many places in Africa, motorbiking in Saskatchewan, Canada and South Africa and finally to the fatal trip to Yemen).

Peter was witty and was not afraid to speak his mind; at our Departmental meetings we now miss his throw-away but very much to-the-point comments and suggestions.

In his teaching and research Peter was somewhat unconventional. He did not follow fashionable trends in research but worked on what interested him most, namely, the geometrisation of physics.

He considered refereeing of research papers to be an important task, writing long reports full of helpful suggestions to the authors. Peter also took his teaching seriously. His courses, often perceived as difficult by the students, were always quite advanced, as if designed to draw the very best from his audience. His view was that it was better to say something new and stimulating to the interested students than show routine steps to the uninterested ones.

The present book grew out of his course on special and general relativity given to our third year students. I first became aware of Peter's approach to relativity when, having taught a similar course before him, I was asked to check his examination questions. While some of them were routine, others demanded deeper thought and when I looked at his solutions I became aware of the merits of his more geometric approach. I was among those who encouraged him to write a book in order to make this approach available to a wider audience.

Peter's book puts an emphasis on geometry in the description of physical phenomena in Minkowski spacetime. In this it emphasises the covariance

properties of the equations of motion, trying as much as possible to avoid working in any particular frame of reference. And the book achieves this aim, probably, more than any other book that I know.

I am very pleased that Springer-Verlag have published Peter's book. The book will not only help many people to understand physics in a more geometrical setting, but also it will be a lasting reminder of our colleague and friend, complementing our personal memories of him.

Wojtek J. Zakrzewski
University of Durham

Preface

This book is **not** meant for the complete beginner in special relativity, **nor** for anyone wanting an account of the numerous and interesting experiments that support the theory. Instead, it is intended to be a description of the geometry of spacetime, and an aid in the creation and development of intuition in four-dimensional Minkowski space. The emphasis on the *geometry* means an emphasis on the *absolutes* which underlie relative descriptions. For example, the Poincaré transformation links different relative sets of coordinates, x^μ , x'^μ , but the underlying absolute is simply a point P in spacetime (the coordinates are the relative descriptions). The deepest understanding, perhaps the only understanding, of relativity and spacetime is in terms of the geometrical absolutes, and this is what the book seeks to develop. Whereas the beginner in special relativity must have help in making the transition between his nonrelativistic view of physics as a time-development in space (his space) to a four-dimensional view of physics as a complete history in spacetime, it is hoped that the reader of this book is ready to study the subject in its final, unified (and beautiful) form.

The mathematical prerequisites for the early chapters of the book are very few, just linear algebra and elementary geometry (done using vectors and a scalar product). For the later chapters multivariable calculus and ordinary differential equations are often needed. No detailed knowledge of the experimental background to relativity is needed, nor any detailed knowledge of electromagnetism, but in both these areas, the more sophistication and sympathy is available for the subjects, the better.

The book aims to cover the most interesting topics requiring special relativity. It is an outgrowth of lectures on special and general relativity given to final year undergraduate students of theoretical physics in the Department of Mathematics. It could be presumed that the students had all had half a dozen or a dozen lectures in earlier years covering the experimental foundations of special relativity and the first, surprising consequences of Einstein's new kinematics. However, the book goes well beyond what was ever taught in practice. Although in a real sense special relativity is the culmination of classical physics, and worthy on that account of detailed study, in the lecture theatre time is limited and the attractions of gravity, with its curved spacetime, become overwhelming. In practice, a natural climax for special

relativity is the definition of the energy tensor (which becomes the source of gravitational curvature) and its use in deriving equations of motion. Some of the more difficult aspects of the energy tensor, and most areas of electromagnetism, were left for self study (in the future). The material in the book, therefore, is partly at an undergraduate level and partly at a postgraduate level.

In the first chapter, Spacetime, the idea of a four-dimensional space having special coordinates (arising from the inertial frames of reference) is developed. An attempt is made to distinguish between the mathematical side of the exposition, where clarity and logic can be expected, and the real-world side, still partly unknown and mysterious, where our understanding advances in a series of temporary world views. The present model is described in natural language (not mathematical); it is a familiar world of clocks and spatial frameworks, but mysteriously without gravity. The mathematics we develop is put into correspondence with this model. The Lorentz transformation and the Poincaré transformation are discussed (as distinct from being postulated, or derived from an artificial starting point). The importance of the lightcone in the theory is exemplified by the way it creates a significant division into regions of the spacetime around any given event. In the whole of the chapter, the emphasis is on spacetime and how we can begin to picture events and processes (and inertial frames, which may be *relative* yet are also physical objects) in it.

In the second chapter, the most important one for building intuition in Minkowski space, vectors in spacetime are defined as transformations of points in spacetime (the geometrical or absolute concept), simply expressed in terms of the inertial frames, which both contribute to the definition and provide the relative expressions of the concept. The scalar product of vectors is constructed to provide the vector expression of the division of spacetime determined by the lightcone. All the famous kinematical effects can be given completely transparent discussions in terms of spacetime diagrams and simple vector geometry.

(The first time I taught the course on which this book is based, I attempted to begin with a discussion of vectors in Minkowski spacetime, without any discussion of spacetime as a manifold. Only a few students found this direct approach attractive and were able to build a useful intuition from it. Student discontent resulted in what is now Chapter 1 to fill in all the background material.)

The third chapter, Asymptotic Momentum Conservation, is devoted to the four-momentum of elementary particles and the relations that follow from the simple idea of equating momentum in the past with momentum in the future. All relations can be expressed in purely geometrical terms. The definition of the centre of momentum frame is particularly simple when it is expressed by its geometrically defining property (rather than in terms of its relation with other, irrelevant frames).

In Chapter 4, covectors and dyadics, which are generalisations of vectors, are defined and their properties developed. The gradient of scalar functions in spacetime is defined as a covector, then converted to a vector, then generalised to the gradient of vector fields and beyond. The concept of volume is discussed, as is the divergence theorem in spacetime.

In Chapter 5, the geometrical formulation of electromagnetism is given. The early sections deal with the decomposition of the field dyadic into relative electric and magnetic fields, and the relation of the different expressions of Maxwell's equations. The geometrical discussion of charge density and three-current is given in terms of a model of charged dust. Conservation of charge then has a visualisable form. The electromagnetism of point particles is begun. Because the consideration of point particles involves delta functions, the topic is technically more difficult and may be omitted at a first reading.

The energy tensor is the subject of Chapter 6. The meaning of its different components is developed with the example of flowing dust. Local conservation of four-momentum is expressed by the vanishing of the divergence of the energy tensor. The equation of motion for flowing, charged dust can be derived from this condition. The general definition of the energy tensor can be developed from a Lagrangian in those cases where the equations of motion can be derived from a variational principle.

A point particle with an accelerating timelike worldline creates some special, peculiarly relativistic problems. It is not self-evident how to define the time development of the rest frame. Two solutions, the Fermi-Walker transported frame and the frame which is boosted from the laboratory, both correspond, nonrelativistically, to the unique "nonrotating" frame of Newtonian mechanics. Yet there is a relative rotation between them, the Thomas precession. These problems and their solutions, and the equation of motion of the spin of a point particle with a magnetic moment, are discussed in Chapter 7*.

After every chapter, but especially the first four, are many exercises and problems which supply lots of opportunity to practise the skills and techniques appropriate to special relativistic geometry. And at the very end of each chapter are listed some references for supplementary reading on particular points. No attempt has been made to provide a complete bibliography.

E.G.P.R
University of Durham
England

* Chapter 7 was incomplete at the time of the author's death, and so is not included in the present volume.

Contents

1. Spacetime	1
1.1 Spacetime is a Four-dimensional Continuum	3
1.2 Aristotelian Spacetime (Pre-relativistic)	4
1.3 Galilean Spacetime	5
1.4 Principles of Special Relativity	7
1.5 Minkowskian Inertial Frames of Reference	8
1.6 Poincaré Transformations	12
1.6.1 Straight Lines	14
1.6.2 Light Rays	16
1.6.3 Units	20
1.6.4 Orientations and Definition of Lorentz Transformations	22
1.6.5 Inverse Lorentz Transformations	22
1.7 Inertial Coordinates in Spacetime	24
1.7.1 Absolute vs Relative Diagrams	24
1.7.2 The Use of Inertial Coordinates	26
1.7.3 Relation of Coordinates for Boosted Frames	26
1.8 Geometrical Relations Between Events	28
1.8.1 Spacetime Interval	28
1.8.2 Invariant Relations	29
1.9 Poincaré Group	31
1.9.1 Subgroup of Translations	31
1.9.2 Rotation Subgroup	32
1.9.3 Boosts Do Not Form a Subgroup	33
1.10 Physical Spacetime Diagrams	34
1.11 Problems	36
References	41
2. Vectors in Spacetime	43
2.1 Translation Vectors in Spacetime	44
2.1.1 Vector Space	45
2.1.2 Addition	46
2.1.3 Multiplication by a Scalar	47
2.1.4 Inertial Basis Vectors	48
2.1.5 Decomposition	49

2.1.6	Transformation of Basis Vectors	49
2.2	Scalar Product of Spacetime Vectors	51
2.3	Classification of Vectors	53
2.3.1	Future-pointing Lightlike Vectors	54
2.3.2	Past-pointing Lightlike Vectors	54
2.3.3	Future-pointing Timelike Vectors	55
2.3.4	Past-pointing Timelike Vectors	55
2.3.5	Spacelike Vectors	55
2.3.6	Zero Vector	56
2.4	The Famous Kinematical Effects	56
2.4.1	Time Dilation	57
2.4.2	The Twin Paradox	59
2.4.3	Length Contraction	62
2.4.4	Addition of Velocities	63
2.4.5	Two Moon Rockets	65
2.4.6	The Problem of Crashing Mirrors	66
2.5	The Generalised Vector Space \mathcal{V}	67
2.6	Proper Time and Concepts of Velocity	68
2.6.1	Spacetime Velocity	69
2.6.2	Proper Time	71
2.6.3	Relative Velocity with Respect to an Inertial Frame... ..	72
2.6.4	General Addition of Velocities Formula	74
2.6.5	Acceleration	75
2.7	Light Rays	76
2.7.1	Lightlike Vectors	76
2.7.2	Harmonic Light	81
2.7.3	Scalar Field Theory for Light	83
2.8	Description of Uniformly Moving Objects	85
2.8.1	Example: Rod Lying in the Direction of Motion	87
2.8.2	Example: Rod at an Angle to the Direction of Motion. ..	88
2.8.3	Example: Parallelogram at Rest in \tilde{K}	88
2.8.4	Example: Parallelepiped at Rest in \tilde{K}	89
2.8.5	Example: A Uniformly Moving Rod Can Appear to Dip ..	89
2.9	Problems	90
	References	98
3.	Asymptotic Momentum Conservation	99
3.1	Particle Momenta	99
3.1.1	Massive Particles	100
3.1.2	Massless Particles	102
3.1.3	Energy and Three-momentum of One Particle With Respect to the Rest Frame of Another Particle	104
3.2	Conservation of Asymptotic Momentum	104
3.3	Three-particle Processes	106
3.4	A Kinematical Function	108

3.5	Compton Effect	109
3.6	Centre-of-momentum Frame	110
3.6.1	Two-particle CM-frame	112
3.7	Threshold Energy for Particle Production	113
3.8	Scattering Formulæ	115
3.8.1	Laboratory frame	116
3.8.2	CM-frame	116
3.9	Problems	119
References	124
4.	Covectors and Dyadics in Spacetime	125
4.1	Covectors in Spacetime	126
4.1.1	Components of a Covector	126
4.1.2	Transformation Law for Components	126
4.1.3	The Dual Space (or Cospace)	127
4.1.4	Cobases and Their Transformation Law	127
4.1.5	The Natural Isomorphism Between \mathcal{V} and \mathcal{V}^*	128
4.1.6	Geometrical Interpretation	128
4.2	Gradient of a Scalar Field	130
4.2.1	Approximation of Scalar Fields and the Covector Gradient	130
4.2.2	Components of the Covector Gradient	131
4.2.3	Vector Gradient	132
4.2.4	Gradient Operators	133
4.2.5	Cobasis as the Covector Gradient of the Inertial Coordinates	133
4.3	Dyadics in Spacetime	133
4.3.1	Linear Transformations as Dyadics	134
4.3.2	Simplest Properties	135
4.3.3	Bases for the Space of Dyadics	135
4.3.4	A Unit Dyadic: The Contravariant Metric	136
4.3.5	A Geometrical Example: Reflection Dyadics	137
4.3.6	Transformation Law for Components	137
4.3.7	Transposed Dyadics and Symmetries	138
4.3.8	Scalar Products	139
4.3.9	Trace	139
4.4	Rotation and Boost Dyadics	140
4.5	Gradient of a Vector Field	141
4.6	Extensions	143
4.7	Dual of an Antisymmetric Dyadic	144
4.7.1	The Definition is Basis-independent	145
4.7.2	Explicit Components of the Dual	146
4.7.3	An Antisymmetric Basis	147
4.7.4	Angular Momentum Dyadic for a Freely Moving Particle	148

4.8	Concept of Volume in Spacetime	149
4.8.1	Dimension Two	149
4.8.2	Unoriented Region.....	149
4.8.3	Oriented Region.....	150
4.8.4	Dimensions Three and Four.....	151
4.8.5	The Common Measure in Minkowski Spacetime	152
4.8.6	Change of Variables Formula.....	153
4.9	Divergence Theorem in Spacetime	154
4.9.1	Integral Form of the Conservation Law	154
4.9.2	Divergence Theorem in Spacetime: Geometrical Expression	156
4.9.3	Divergence Theorem: Analytical Expression.....	157
4.10	Problems	158
	References	164
5.	Electromagnetism	165
5.1	Maxwell's Equations	165
5.1.1	Verification of Maxwell's Equations	167
5.2	Transformation of Electric and Magnetic Fields	168
5.3	Example: An Infinite Line of Charge	169
5.3.1	Fields and Sources with Respect to the Rest Frame K	170
5.3.2	Fields and Sources with Respect to K'	171
5.4	Vector Potential.....	172
5.4.1	Alternative Form for the Homogeneous Maxwell Equation	173
5.4.2	An Explicit Vector Potential	173
5.4.3	Lorentz Condition	174
5.5	Electric Current Density	174
5.5.1	Charged Dust	175
5.5.2	Density of Charge	176
5.5.3	Flux of Charge.....	177
5.5.4	Conservation of Charge	178
5.5.5	Conservation of Charge Along a Worldline	180
5.6	Point Particle: A Singular Source	180
5.6.1	The Intrinsic Variables	182
5.6.2	Electromagnetic Field for a Point Charge.....	185
5.6.3	Electric and Magnetic Fields in the Retarded Rest Frame	186
5.6.4	Electric and Magnetic Fields in the Laboratory	186
5.6.5	Maxwell's Equations for a Point Charge.....	187
5.6.6	The Region Off the Worldline: Empty Space	188
5.6.7	The Electromagnetic Field as a Distribution	188
5.6.8	Change of Variables in Spacetime Integrals	189
5.6.9	Maxwell's Equations Along the Worldline	190
5.7	Plane Waves	191

5.7.1	Plane Polarised Waves	192
5.7.2	Circularly Polarised Waves	193
5.7.3	Change of Basis: A Boost in the Direction of Propagation	193
5.7.4	Change to a General Moving Frame	194
5.8	Problems	195
	References	200
6.	The Energy Tensor	201
6.1	The Energy Tensor for Dust	202
6.2	The Energy Tensor in General	203
6.2.1	Conservation of Four-momentum	204
6.3	The Variational Principle	206
6.3.1	Action for the Electromagnetic Field	208
6.3.2	Action for a Charged Particle in an External Field	209
6.3.3	Action for Charged Particles Interacting Electromagnetically	210
6.4	Noninertial Coordinates	211
6.4.1	New Basis Vectors	211
6.4.2	New Components	212
6.5	Construction of the Energy Tensor	213
6.5.1	First Example: Energy Tensor for the Free Scalar Field	218
6.5.2	The Total Energy Tensor Has Zero Divergence	219
6.5.3	Second Example: Energy Tensor for the Electromagnetic Field	219
6.5.4	Third Example: Energy Tensor for a Point Particle	220
6.6	Energy in the Electromagnetic Field	221
6.6.1	Four-momentum in a Plane Wave	222
6.6.2	Radiation From an Accelerating Point Charge	223
6.7	Equations of Motion for Charged Dust	224
6.7.1	Uncharged Incoherent Dust	224
6.7.2	Charged Dust	226
6.7.3	Frame-dependent Equation of Motion for Charged Dust	227
6.8	Perfect Fluid	228
6.8.1	Equations of Motion	229
6.8.2	First Law of Thermodynamics	229
6.8.3	Label Space	230
6.8.4	Lagrangian for a Perfect Fluid	232
6.9	Problems	235
	References	239
	Index	241