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Local Multipliers of C^*-Algebras

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Preface

Many problems in operator theory lead to the consideration of operator equations, either directly or via some reformulation. More often than not, however, the underlying space is too 'small' to contain solutions of these equations and thus it has to be 'enlarged' in some way. The Berberian–Quigley enlargement of a Banach space, which allows one to convert approximate into genuine eigenvectors, serves as a classical example. In the theory of operator algebras, a C^*-algebra A that turns out to be small in this sense traditionally is enlarged to its (universal) enveloping von Neumann algebra A''. This works well since von Neumann algebras are in many respects richer and, from the Banach space point of view, A'' is nothing other than the second dual space of A. Among the numerous fruitful applications of this principle is the well-known Kadison–Sakai theorem ensuring that every derivation δ on a C^*-algebra A becomes inner in A'', though δ may not be inner in A.

The transition from A to A'' however is not an algebraic one (and cannot be since it is well known that the property of being a von Neumann algebra cannot be described purely algebraically). Hence, if the C^*-algebra A is small in an algebraic sense, say simple, it may be inappropriate to move on to A''. In such a situation, A is typically enlarged by its multiplier algebra $M(A)$. It has emerged that this process is much more than a clever way of adding an identity to A, for $M(A)$ became an indispensable tool in dealing with Hilbert C^*-modules, extensions, K- and KK-theory, and, most recently, the classification of C^*-algebras. And, taking up the above example, another famous result due to Sakai states that every derivation of a simple C^*-algebra will become inner in its multiplier algebra.

When encountering a C^*-algebra A with non-trivial ideal structure, it is thus natural to think of an enlargement of A that will contain all the multipliers of closed ideals of A and to use it in a similar way as $M(A)$ in questions on operators on A compatible with the ideal structure. In the extreme case of a commutative C^*-algebra A, the multipliers of closed ideals correspond to complex-valued functions that are bounded and continuous on open subsets of the structure space of A only, i.e., they are 'locally' defined. This yields the name *local multiplier algebra* $M_{\mathrm{loc}}(A)$ for this type of enlargement of A; the same name is also used in the general case. Under the name 'essential multipliers', $M_{\mathrm{loc}}(A)$ was first used by Elliott and Pedersen in the mid 1970's, but it was not studied further until the late 1980's. Then its investigation was taken up again by the present authors, who were mainly motivated by its possible applications in operator theory on C^*-algebras. The purpose of this book is to set out the results of this study in a comprehensive frame.

In the case of a simple C^*-algebra A, of course, $M(A)$ equals $M_{\mathrm{loc}}(A)$, so nothing new is added (and nothing already gained is lost). But, if we move on to the next complicated situation of a prime C^*-algebra A, many results on operators on A take the same form as in the simple case, although $M(A)$ no longer coincides with $M_{\mathrm{loc}}(A)$. It turns out that this is due to the fact that the centres $Z(M(A))$ and $Z(M_{\mathrm{loc}}(A))$ agree (and are isomorphic to $\mathbf{C}$). An arbitrary C^*-algebra A with this property will be called *boundedly centrally closed*. It is this class of C^*-algebras which are 'large' and best behaved in operator theory. Since they include von Neumann algebras, many results previously obtained in that setting are now seen in a new perspective, often with novel, simpler proofs.

In undertaking this endeavour, we were guided by the concepts of generalised rings of quotients of semiprime rings R, in particular, the Kharchenko symmetric ring of quotients $Q_s(R)$ and its centre, the extended centroid. Several results valid for centrally closed rings (those for which the centroid $Z(M(R))$ and the extended centroid $Z(Q_s(R))$ coincide) find their analogues in results for boundedly centrally closed C^*-algebras. In fact, there is a natural relation between $Q_s(A)$ and $M_{\mathrm{loc}}(A)$ for every C^*-algebra A: $Q_s(A)$ is the central localisation of a dense *-subalgebra $Q_b(A)$ of $M_{\mathrm{loc}}(A)$, and it is often rather expedient, and sometimes inevitable, to work within the surrounding algebraic framework $Q_s(A)$ first, and, in a second step, to find the solution to a problem inside the bounded part $Q_b(A)$. As a result, we decided to provide also the necessary algebraic requisites in a (mostly) self-contained way.

In this book, the foundations of the concept of local multipliers of C^*-algebras are carefully laid down, the results previously published in research journals are put into a coherent and comprehensive frame, and many unpublished new results and applications in operator theory are obtained. We hope to address both a C^*-algebraic audience by providing a new powerful tool as well as those with mainly algebraic interests by illustrating applications of algebraic methods in analysis.

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