

# LAGRANGE-TYPE FUNCTIONS IN CONSTRAINED NON-CONVEX OPTIMIZATION

# **Applied Optimization**

**Volume 85**

---

*Series Editors:*

**Panos M. Pardalos**  
*University of Florida, U.S.A.*

**Donald W. Hearn**  
*University of Florida, U.S.A.*

# LAGRANGE-TYPE FUNCTIONS IN CONSTRAINED NON-CONVEX OPTIMIZATION

ALEXANDER RUBINOV  
School of Information Technology  
and Mathematical Sciences,  
University of Ballarat,  
Victoria, Australia

XIAOQI YANG  
Department of Applied Mathematics  
Hong Kong Polytechnic University,  
Hong Kong, China



Springer-Science+Business Media, B.V.

**Library of Congress Cataloging-in-Publication**

Rubinov, Alexander/ Yang, Xiaoqi  
Lagrange-type Functions in Constrained Non-convex Optimization  
ISBN 1-4020-7627-4

ISBN 978-1-4613-4821-4      ISBN 978-1-4419-9172-0 (eBook)  
DOI 10.1007/978-1-4419-9172-0

---

Copyright © 2003 by Springer Science+Business Media Dordrecht  
Originally published by Kluwer Academic Publishers

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photo-copying, microfilming, recording, or otherwise, without the prior written permission of the publisher, with the exception of any material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work.

Permissions for books published in the USA: [permissions@wkap.com](mailto:permissions@wkap.com)

Permissions for books published in Europe: [permissions@wkap.nl](mailto:permissions@wkap.nl)

*Printed on acid-free paper.*

# Contents

Preface	ix
Acknowledgments	xiii
1. INTRODUCTION	1
1.1 Introduction and motivation	1
1.2 Duality	6
1.3 Mathematical tools	10
1.4 Notation	12
2. ABSTRACT CONVEXITY	15
2.1 Abstract convexity	15
2.1.1 Definitions and preliminary results	15
2.1.2 Fenchel–Moreau conjugacy and subdifferential	18
2.1.3 Abstract convex at a point functions	20
2.1.4 Subdifferential	23
2.1.5 Abstract convex sets	24
2.2 Increasing positively homogeneous (IPH) functions	25
2.2.1 IPH functions: definitions and examples	25
2.2.2 IPH functions defined on $\mathbb{R}_{++}^2$ and $\mathbb{R}_+^2$	26
2.2.3 Associated functions	32
2.2.4 Strictly IPH functions	41
2.2.5 Multiplicative inf-convolution	45
3. LAGRANGE-TYPE FUNCTIONS	49
3.1 Conditions for minimum in terms of separation functions	49
3.1.1 Problem $P(f, g)$ and its image space	49
3.1.2 Optimality conditions through the intersection of two sets	51

3.1.3	Optimality conditions via separation functions: linear separation	53
3.1.4	Optimality conditions via separation functions: general situation	56
3.1.5	Perturbation function	61
3.1.6	Lower semicontinuity of perturbation function	62
3.2	Lagrange-type functions and duality	66
3.2.1	Convolution functions	66
3.2.2	Lagrange-type functions	68
3.2.3	Lagrange-type functions with multipliers	69
3.2.4	Linear outer convolution function	71
3.2.5	Penalty-type functions	72
3.2.6	Auxiliary functions for methods of centers	73
3.2.7	Augmented Lagrangians	73
3.2.8	Duality: a list of the main problems	76
3.2.9	Weak duality	78
3.2.10	Problems with a positive objective function	81
3.2.11	Giannessi scheme and RWS functions	82
3.3	Zero duality gap	85
3.3.1	Zero duality gap property	85
3.3.2	Special convolution functions	87
3.3.3	Alternative approach	90
3.3.4	Zero duality gap property and perturbation function	92
3.4	Saddle points	96
3.4.1	Weak duality	96
3.4.2	Saddle points	96
3.4.3	Saddle points and separation	99
3.4.4	Saddle points, exactness and strong exactness	103
4.	PENALTY-TYPE FUNCTIONS	109
4.1	Problems with a single constraint	109
4.1.1	Reformulation of optimization problems	109
4.1.2	Transition to problems with a single constraint	110
4.1.3	Optimal value of the transformed problem with a single constraint	113
4.2	Penalization of problems with a single constraint based on IPH convolution functions	115
4.2.1	Preliminaries	115
4.2.2	Class $\mathcal{P}$	117

4.2.3	Modified perturbation functions	118
4.2.4	Weak duality	120
4.2.5	Associated function of the dual function	120
4.2.6	Zero duality gap property	123
4.2.7	Zero duality gap property (continuation)	128
4.3	Exact penalty parameters	129
4.3.1	The existence of exact penalty parameters	129
4.3.2	Exact penalization (continuation)	131
4.3.3	The least exact penalty parameter	134
4.3.4	Some auxiliary results. Class $B_X$	137
4.3.5	The least exact penalty parameter (continuation)	141
4.3.6	Exact penalty parameters for function $s_k$	143
4.3.7	The least exact penalty parameter for function $s_k$	146
4.3.8	Comparison of the least exact penalty parameters for penalty functions generated by $s_k$	148
4.3.9	Lipschitz programming and penalization with a small exact penalty parameter	153
4.3.10	Strong exactness	155
4.4	The least exact penalty parameters via different convolution functions	156
4.4.1	Comparison of exact penalty parameters	156
4.4.2	Equivalence of penalization	159
4.5	Generalized Lagrange functions for problems with a single constraint	161
4.5.1	Generalized Lagrange and penalty-type functions	161
4.5.2	Exact Lagrange parameters: class $\mathcal{P}_*$	163
4.5.3	Zero duality gap property for generalized Lagrange functions	164
4.5.4	Existence of Lagrange multipliers and exact penalty parameters for convolution functions $s_k$	168
5.	AUGMENTED LAGRANGIANS	173
5.1	Convex augmented Lagrangians	173
5.1.1	Augmented Lagrangians	173
5.1.2	Convex augmenting functions	176
5.2	Abstract augmented Lagrangians	177
5.2.1	Definition of abstract Lagrangian	178
5.2.2	Zero duality gap property and exact parameters	179
5.2.3	Abstract augmented Lagrangians	181
5.2.4	Augmented Lagrangians for problem $P(f, g)$	185

5.2.5	Zero duality gap property for a class of Lagrange-type functions	188
5.3	Level-bounded augmented Lagrangians	190
5.3.1	Zero duality gap property	190
5.3.2	Equivalence of zero duality gap properties	196
5.3.3	Exact penalty representation	201
5.4	Sharp augmented Lagrangians	206
5.4.1	Geometric interpretation	206
5.4.2	Sharp augmented Lagrangian for problems with a single constraint	210
5.4.3	Dual functions for sharp Lagrangians	212
5.5	An approach to construction of nonlinear Lagrangians	215
5.5.1	Links between augmented Lagrangians for problems with equality and inequality constraints	215
5.5.2	Supergradients of the dual function	219
6.	OPTIMALITY CONDITIONS	221
6.1	Mathematical preliminaries	222
6.2	Penalty-type functions	227
6.2.1	Differentiable penalty-type functions	227
6.2.2	Nondifferentiable penalty-type functions	232
6.3	Augmented Lagrangian functions	244
6.3.1	Proximal Lagrangian functions	244
6.3.2	Augmented Lagrangian functions	249
6.4	Approximate optimization problems	252
6.4.1	Approximate optimal values	252
6.4.2	Approximate optimal solutions	260
7.	APPENDIX: NUMERICAL EXPERIMENTS	265
7.1	Numerical methods	265
7.2	Results of numerical experiments	268
	Index	285



# Preface

Lagrange and penalty function methods provide a powerful approach, both as a theoretical tool and a computational vehicle, for the study of constrained optimization problems. However, for a nonconvex constrained optimization problem, the classical Lagrange primal-dual method may fail to find a minimum as a zero duality gap is not always guaranteed. A large penalty parameter is, in general, required for classical quadratic penalty functions in order that minima of penalty problems are a good approximation to those of the original constrained optimization problems. It is well-known that penalty functions with too large parameters cause an obstacle for numerical implementation. Thus the question arises how to generalize classical Lagrange and penalty functions, in order to obtain an appropriate scheme for reducing constrained optimization problems to unconstrained ones that will be suitable for sufficiently broad classes of optimization problems from both the theoretical and computational viewpoints.

Some approaches for such a scheme are studied in this book. One of them is as follows: an unconstrained problem is constructed, where the objective function is a convolution of the objective and constraint functions of the original problem. While a linear convolution leads to a classical Lagrange function, different kinds of nonlinear convolutions lead to interesting generalizations. We shall call functions that appear as a convolution of the objective function and the constraint functions, *Lagrange-type functions*. We observe that these functions naturally arise as a result of a nonlinear separation of the image set of the problem and a cone in the image-space of the problem under consideration. The class of Lagrange-type functions includes also augmented Lagrangians, corresponding to the so-called canonical dualizing parameterization. However, augmented Lagrangians constructed by means of some general dualizing parameterizations cannot be included in this scheme. We consider them separately.

In a recent corner-stone book [102], an elegant duality theory was developed for an augmented Lagrangian with a convex augmenting function for (nonconvex) optimization problems. However, convexity of augmenting functions for augmented Lagrangians and of convolution functions for Lagrange-type functions sometimes is a restrictive assumption. The following example confirms this: while classical exact penalty functions may not exist for mathematical programs with complementarity constraints, a class of lower order nonconvex and nonsmooth exact penalty functions can be established, see [80]. Our re-

sults also confirm that concave convolution functions are better than convex ones for some nonconvex optimization problems, e.g., for a concave minimization problem over a polyhedron set. We study a very general class of abstract Lagrangians, which includes those defined by convex augmenting functions, and level-bounded augmenting functions, as special cases.

The purpose of this book is to provide a systematic examination of Lagrange-type functions and augmented Lagrangians. We will study these functions from three aspects: weak duality, zero duality gap property and the existence of an exact penalty parameter. Weak duality allows one to estimate a global minimum, zero duality gap property allows one to reduce the constrained optimization problem to a sequence of unconstrained problems, and the existence of an exact penalty parameter allows one to solve only one unconstrained problem. By applying Lagrange-type functions, we are able to establish a zero duality gap property for nonconvex constrained optimization problems under a coercive condition. We show that the zero duality gap property is equivalent to the lower semi-continuity of a perturbation function.

The numerical implementation of penalty functions requires the existence of a fairly small exact penalty parameter. This is very important for all local methods and some global methods of Lipschitz programming, otherwise ill-conditioning may occur [34, 42]. We use so-called IPH (increasing positively homogeneous) functions for the convolution of the objective and the constraint functions. Special attention is in particular paid to problems with a single constraint, as optimization problems with multiple constraints can be reduced to such a problem by convoluting all constraints into a single one. For a kind of  $k$ th penalty functions we are able to obtain an analytic expression for the least exact penalty parameter. By virtue of this expression we show that the least exact penalty parameter of an  $k$ th power penalty function can be diminished if  $k$  is small enough. This result leads to a certain reformulation of the initial problem, which allows us to develop and implement a new type of penalty-type functions. These functions can be applied for concave minimization, where the classical penalty function fails.

The outline of the book is as follows.

In Chapter 1, we present motivation for studying Lagrange-type functions, and discuss main questions related to Lagrange-type functions and augmented Lagrangians.

In Chapter 2, we present some auxiliary results related to abstract convexity and theory of IPH functions. The reader can find there a description of the technique, which is used for examination of the zero duality gap property and penalty-type functions for problems with a single constraint.

In Chapter 3, we develop a general scheme of Lagrange-type functions, which is based on a separation of certain sets in the image-space of the problem. We use an elegant idea of Giannessi's as a starting point of our research. Some general

results related to weak duality and zero duality gap property are established. The theory of saddle points for Lagrange-type functions, which are linear with respect to the objective function, can also be found there.

In Chapter 4, we consider penalty-type functions for problems with a single constraint using an IPH convolution function. We estimate the least exact penalty parameter for various problems and investigate which reformulation of constrained optimization problems is better from the applications viewpoint. We aim to obtain estimates of parameters, which appear in our approach, and that of the least exact penalty parameter for several classes of problems.

In Chapter 5, we study the zero duality gap property and exactness for various augmented Lagrangian functions, including abstract augmented Lagrangian, level-bounded augmented Lagrangian and sharp augmented Lagrangian.

In Chapter 6, we provide a systematic convergence analysis of optimality conditions of nonlinear penalty-type functions and augmented Lagrangian functions to that of the original constrained optimization problem. The study of approximate solutions and optimal values in terms of Lagrange-type functions is also presented.

Appendix (Chapter 7) contains results of numerical experiments, which confirm that the proposed new nonlinear penalty function works well for some problems of non-convex optimization, including the minimization of a concave function subject to linear constraints.

Some preliminary results on the nonlinear Lagrange-type functions have been included as book chapters in the books ‘Abstract Convexity and Global Optimization’ [105] by the first author and ‘Duality in Optimization and Variational Inequalities’ [54] by the second author and his collaborator.

Alex Rubinov

Xiaoqi Yang

# Acknowledgments

Some sections of this book contain results that have been obtained by the authors in collaboration with M. Andramonov, A. Bagirov, Yu. Evtushenko, R. Gasimov, J. Giri, B. Glover, C.J. Goh, X.X. Huang, D. Li, A. Uderzo and V. Zhadan. The authors are very grateful to all these colleagues.

The authors feel deeply grief when W. Oettli, a good colleague, pass away on December 2000, while his joint paper [89] with the second author has made significant contribution to the topic of this book.

We are also very thankful to J. Dutta and X.X. Huang, who read carefully parts of the book and suggested some improvements.

We are thankful to the Australian Research Council and the Research Grants Council of Hong Kong for their financial supports of this project.

The idea to write this book was supported by J. Martindale, a senior editor of Kluwer Academic Publishers and by Professor P. Pardalos, Managing Editor of the series *Nonconvex Optimization with Applications*, in which this book will be published. We are very thankful to them for their continuous support.