Low Complexity MIMO Detection

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To our families and friends

Foreword

What is the most important emerging technology leading to high data rate wireless services? With scarce wireless spectrum, the use of multiple antennas is becoming the key foundation to achieve the requirement. My colleagues, Prof. Bai and Prof. Choi have worked on this topic for many years. They have made good achievements and published a number of papers within this topic. In this book, they share their key findings.

With signal detection methods now representing a key application of signal processing methods to communication systems, this book provides a range of important techniques for signal detection when multiple transmitted and received signals are available. In this book, various optimal and suboptimal signal detection methods are explained in the context of multiple-input multiple-output (MIMO) systems, including list decoding and lattice reduction (LR)-aided detection, while various user selection schemes are also discussed within multiuser systems. Those techniques are then analyzed using performance analysis tools.

With a carefully balanced blend of theoretical elements and applications, this book is ideal for both graduate students and practicing engineers in wireless communications. All the techniques introduced in this book are quite new. Furthermore, this book makes an easy-to-follow presentation from the elementary to the profound level.

Beijing

Quan Yu
Academician
Chinese Academy of Engineering

Preface

In order to improve the spectral efficiency in wireless communications, multiple antennas are employed at both transmitter and receiver sides, where the resulting system is referred to as the multiple-input multiple-output (MIMO) system. In MIMO systems, it is usually required to detect signals jointly as multiple signals are transmitted through multiple signal paths between the transmitter and the receiver. This joint detection becomes the MIMO detection.

The MIMO detection can be performed by an exhaustive search method for the maximum likelihood (ML) detection. Unfortunately, although this method provides the optimal performance, it is impractical for a number of real systems since its complexity grows exponentially with the number of transmit antennas. For the case of MIMO channels in cellular systems where the transmitter is a base station and the receiver is a mobile terminal, since the receiver usually has a limited computing power for symbol detection, the use of ML detection based on an exhaustive search or those with high computational complexity becomes impossible. To avoid this prohibitively high computational complexity, computationally efficient suboptimal MIMO detection methods are investigated, including linear detectors that take the signals from the other antennas as the interference; but, poor performance is expected due to a high date error rate. Therefore, it is desired to develop MIMO detection methods that have near optimal performance as well as low computational complexity. In this book, we attempt to explain such low complexity MIMO detectors.

So far, there are many existing books related to MIMO systems. To be different from those books, our book focuses on low complexity MIMO symbol detection itself. Although our book is very specific, we have adopted an easy-to-follow presentation from the elementary to the profound level. Furthermore, we include a number of recent research outcomes that are also useful for those experts in this area.

Our group has worked on the design of low complexity MIMO detection for many years and has produced various new results on low complexity MIMO detection with the ideas of list decoding and lattice basis reduction. In addition, as an extension, multiuser MIMO and the corresponding strategies are also investigated. This book includes not only our research outcomes but also other recent research outcomes that could be very useful to practitioners and postgraduate students who want to learn new outcomes of low complexity MIMO detectors in the field of wireless communications.

This book systematically introduces the signal detection in MIMO systems. It has been written for the reader who wants to become an expert from a beginner in the field of MIMO detection. In addition, it is suitable for postgraduate students who have some fundamental knowledge of wireless communications, and for R&D personnel who works in MIMO area.

Beijing Lin Bai
Swansea Jinho Choi

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Acronyms

APP	A posteriori probability
APRP	A priori probability
AWGN	Additive white Gaussian noise
AS	Antenna subset
BER	Bit error rate
BPSK	Binary phase shift keying
BS	Base station
cdf	Cumulative density function
CLLL	Complex-valued LLL
CMs	Complex multiplications
CRC	Column reordering criteria
CRIS	Column reordering index set
CSCG	Circular symmetric complex Gaussian
CSI	Channel state information
DFE	Decision feedback equalizer
DMT	Diversity multiplexing trade-off
DRC	Dimension reduction condition
EP-CRC	Error probability based CRC
flops	Floating point operation
GS	Gram–Schmidt
GSD	Generalized sphere decoding
ISI	Intersymbol interference
LAPPR	Logarithms of a posteriori probability ratios
LBR	Lattice basis reduced
LLL	Lenstra–Lenstra–Lovász
LLR	Log-likelihood ratio
LR	Lattice reduction
LRG	LR-based greedy
MAP	Maximum a posteriori probability
MD	Max–min diagonal term
MDist	Max–min distance

ME	Max–min eigenvalue
MIMO	Multiple-input multiple-output
ML	Maximum likelihood
MMI	Maximum mutual information
MMMSE	Min–max mean square error
MMSE	Minimum mean square error
MSE	Mean square error
OD-CRC	Orthogonality deficiency-based CRC
ODR	Optimal decision region
OFDM	Orthogonal frequency division multiplexing
PAM	Pulse amplitude modulation
pdf	Probability density function
PDR	Probability of dimension reduction
PEP	Pairwise error probability
PVC	Prevoting cancellation
PVS	Postvoting vector selection
QAM	Quadrature amplitude modulation
SDMA	Space division multiple access
SER	Symbol error rate
SIC	Successive interference cancellation
SINR	Signal to interference plus noise ratio
SISO	Single-input single-output
SNR	Signal-to-noise ratio
SSE	Sum of squared error
SVP	Shortest vector problem
TSD-CR	Tree search decoder-column reordering
UBLR	Updated basis LR
UBLRG	UBLR-based greedy
UILS	Underdetermined integer least squares
V-BLAST	Vertical Bell laboratories layered space–time
VLSI	Very large scale integration
ZF	Zero forcing

Notations

$\mathbf{A}/\mathbf{a}$	(Boldface upper/lower letters) complex-valued matrix/vector
$\mathbf{A}_\mathrm{r}/\mathbf{a}_\mathrm{r}$	(Boldface upper/lower letters) real-valued matrix/vector
$\mathbf{A}^\mathrm{T}, \mathbf{A}^\mathrm{H}, \mathbf{A}^\dagger$	Transpose, Hermitian transpose, Pseudo inverse, respectively
$[\mathbf{A}]_{p,q}$	The $(p,\ q)$th element of $\mathbf{A}$
$\mathbf{A}(a : b, c : d)$	The submatrix of $\mathbf{A}$ with the elements obtained from rows $a, \ldots, b$ and columns $c, \ldots, d$
$\mathbf{A}(:, n)$	The nth column vector of $\mathbf{A}$
$\mathbf{A}(n, :)$	The nth row vector of $\mathbf{A}$
$\mathrm{Tr}(\mathbf{A})$	The trace operation of a square matrix $\mathbf{A}$
$\det(\mathbf{A})$	Determinant of matrix $\mathbf{A}$
$\mathrm{adj}(\mathbf{A})$	Adjoint of matrix $\mathbf{A}$
$\mathcal{D}(\mathbf{A})$	Length of the shortest nonzero vector of the lattice generated by $\mathbf{A}$
$\lambda_{\min}(\mathbf{A})$	Minimum eigenvalue of $\mathbf{A}$
$\mathcal{L}(\mathbf{A})$	Lattice generated by $\mathbf{A}$
$E[\cdot]$	Statistical expectation
$\Re(\cdot),\ \Im(\cdot)$	Real and imaginary parts
$< \mathbf{a}, \mathbf{b} >$	Inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathcal{CN}(\mathbf{m}, \mathbf{C})$	Complex Gaussian vector distribution with mean $\mathbf{m}$ and covariance $\mathbf{C}$
$\log(\cdot)$	Natural logarithm
$\mathbf{0}$	Matrix with all entries of 0
$\\| \cdot \\|$	2-norm
$\\|\cdot\\|_\mathrm{F}$	The Frobenius norm
$\lceil\beta\rfloor$	The nearest integer to β
$\lfloor\beta\rfloor$	The closest integer which is smaller than β
$\vert\beta\vert$	Absolute value of scalar β
$\setminus$	Set minus

$\mathbf{I}_n$	An $n \times n$ identity matrix
$\{k_{(1)}, k_{(2)}, \ldots\}$	The collection set of $k_{(1)}, k_{(2)}, \ldots$
$\operatorname{erfc}(x)$	Complementary error function of x, i.e., $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-z^2} \mathrm{d}z$
$\{\exists x : f(x)\}$	There is at least one x such that a function of x, $f(x)$, is true
$\mathbb{Z}$	Set of integer numbers