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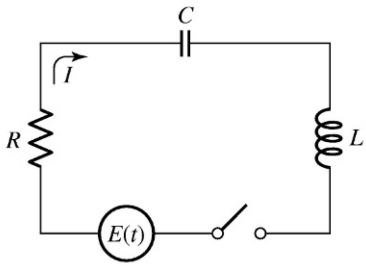
J. David Logan

A First Course in Differential Equations

Second Edition



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To my son David

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Preface to the Second Edition

The goal of this book is the same as the goal of the original edition, namely, to present a one-semester, brief treatment of the key ideas, models, and solution methods in elementary differential equations. As in the first edition, there remains an intimate connection between the mathematics and applications. There are many excellent texts on differential equations designed for the standard sophomore course, but, in spite of the fact that most courses are one semester in length, they have evolved into calculus-like presentations that include a large collection of methods and applications, packaged with student manuals, and Web-based notes, projects, and supplements. All of this comes in several hundred pages of text with busy formats. Many students do not have the time or desire to read voluminous texts and explore Internet supplements. Therefore, the format of this text is different; it is more concise. I have tried to write to the point with plain language. Many worked examples and exercises are included. A student who works through this primer will have the tools to go to the next level in applying differential equations to more difficult problems in engineering, science, and applied mathematics. It gives some instructors who want more concise coverage an alternative to existing texts.

There are a few substantial changes to this new edition. Many users of the text, including several of my colleagues at Nebraska, have contacted me with suggestions and corrections, and I have tried to address their comments. The typographical errors have been corrected, there are more routine exercises designed for practice, there are more examples worked out in the text, and explanations have been expanded in places where the exposition was too terse. One major change is the reorganization of the first two chapters; for example, separation of variables is introduced much earlier in the book, and linear equations are solved using integrating factors rather than variation of parameters.

Second, the last two chapters, on systems of differential equations, have been divided into three. This gives the instructor more flexibility in covering systems. Chapter 5 gives a gentle introduction to systems in general, both linear and nonlinear, without going into depth or matrix methods. Therefore, an instructor desiring only to spend a short amount of time on systems can cover most of Chapter 5. An instructor wishing to spend a substantial portion of the course on systems can find linear systems discussed in detail in Chapter 6, including matrices and eigenvalues, and nonlinear systems in Chapter 7, including linearization and nonlinear dynamics.

As in the first edition, there is flexibility regarding use of software. Students may use a calculator or a computer algebra system to solve some problems numerically or symbolically, and templates of MATLAB® and Maple programs and commands are given in an appendix. The instructor can include as much of this, or as little of this, as he or she desires, or easily adapt the text to other systems, such as Maple *Mathematica*, R, or whatever.

For many years I have taught this material to students who have had a standard three-semester calculus sequence. It was well received by those who appreciated having a small definitive parcel of material to learn. Moreover, this text gives students the opportunity to start reading mathematics at a slightly higher level than they experienced in precalculus and calculus. Therefore, the text can begin a bridge in their progress to study more advanced material at the junior–senior level, where books leave more to the reader and are not packaged in elementary wordy formats.

Chapters 1, 2, 3, 5, 6, and 7 should be covered in order. They provide a route to geometric understanding, the phase plane, and the qualitative ideas that are important in differential equations. Included are the usual treatments of separable and linear first-order equations, along with second-order linear homogeneous and nonhomogeneous equations. There are many applications to ecology, physics, engineering, and other areas. These topics give students basic skills in the subject. Chapter 4, on Laplace transforms, may be covered at any time after Chapter 3, or even omitted. Always an issue in teaching differential equations is how much linear algebra to cover. In two extended sections in Chapter 6 we introduce a moderate amount of matrix theory, including the solution of linear systems, determinants, and the eigenvalue problem. In spite of the book's brevity, it still contains more material than can be comfortably covered in a single, three-hour, semester course. Chapters 1–5 make a good introductory 3-hour course.

The sections in the book, and entries in the table of contents, marked with an asterisk are optional and may be omitted. At the end of the text are practice examination problems and solutions to most of the even exercises. The solutions to the exercises vary from a hint, a brief answer, or a detailed outline to the solution.

I greatly welcome suggestions, comments, and corrections. Contact information is on my web site: `http://www.math.unl.edu/~dlogan`, where additional items may be found.

Finally, I would like to thank my editor Kaitlin Leach at Springer for her enthusiastic support and efficient management of this project. And I greatly appreciate the suggestions passed along to me from the many users of the first edition.

I affectionately dedicate this book to my son David. His unique and insightful perspectives on life, learning, teaching, and scholarship have challenged and influenced me in myriad and remarkable ways. Thank you, David.

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To the Student

What is a course in differential equations about? Here are some informal preparatory remarks to give you some sense of the subject before we take it up seriously. This section should not be skipped!

You are familiar with algebra problems and solving algebraic equations. For example, the solutions to the quadratic equation

$$x^2 - x = 0$$

are easily found to be $x = 0$ and $x = 1$, which are numbers. A differential equation (often abbreviated *DE*) is another type of equation where the unknown is not a number, but a function. We call the unknown function $u(t)$ and think of it as a function of time. Simply, a DE is an equation that relates the unknown function $u(t)$ to some of its derivatives, which, of course, are not known either. A simple example of a DE is

$$u'(t) = u(t),$$

where $u'(t)$ denotes the derivative of $u(t)$.[1] We ask what function $u(t)$ solves this equation. That is, what function $u(t)$ has a derivative that is equal to itself? From calculus you know that one such function is $u(t) = e^t$, the exponential function. We say this function is a solution of the DE, or it solves the DE. Is it the only solution? If you try $u(t) = Ce^t$, where C is any constant whatsoever, you will also find it is a solution. It is generally true that differential equations have many solutions; fortunately these solutions are quite similar, and the fact that there are many allows some flexibility in imposing other desired conditions. For example, among them we can try to find a solution that passes through a given point (t_0, u_0) in the tu plane.

[1] We mostly use the "prime" notation for the derivative.

The preceding DE was very simple and we could guess the answer from our calculus knowledge. But, unfortunately (or, fortunately!), differential equations are usually more complicated. Consider, for example, the DE

$$u''(t) + 2u'(t) + 2u(t) = 0.$$

This equation involves an unknown function $u(t)$ and both its first and second derivatives. In words, we seek a function for which its second derivative, plus twice its first derivative, plus twice the function itself, is zero. Now can you quickly guess a function $u(t)$ that solves this equation? It is not likely. One solution is

$$u(t) = e^{-t} \cos t.$$

Another is

$$u(t) = e^{-t} \sin t$$

Let's check this last one by using the product rule and calculating its derivatives:

$$\begin{aligned} u(t) &= e^{-t} \sin t, \\ u'(t) &= e^{-t} \cos t - e^{-t} \sin t, \\ u''(t) &= -e^{-t} \sin t - 2e^{-t} \cos t + e^{-t} \sin t. \end{aligned}$$

Then, it is easy to see that

$$u''(t) + 2u'(t) + 2u(t) = 0.$$

So it works! The function $u(t) = e^{-t} \sin t$ solves the equation $u''(t) + 2u'(t) + 2u(t) = 0$. You should check right now that the other function $u(t) = e^{-t} \cos t$ works as well. In fact, if you multiply each of these solutions by any constant and add the result to get

$$u(t) = Ae^{-t} \sin t + Be^{-t} \cos t$$

you will find that it is a solution as well, regardless of the values of the constants A and B. To repeat, differential equations have lots of solutions.

Partly, the subject of differential equations is about learning techniques, or methods, for finding solutions.

Why differential equations? Why are they so important to deserve an entire course of study? Well, differential equations arise naturally as *models* in areas of science, engineering, economics, and lots of other subjects. Physical systems, biological systems, and economic systems; all these are marked by change, or dynamics. Differential equations model real-world systems by describing how they change. The unknown function $u(t)$ could be the current in an electrical circuit, the concentration of a chemical undergoing reaction, the population of an animal species in an ecosystem, or the demand for a commodity in a micro-economy. Differential equations represent laws that dictate change, and the unknown $u(t)$, for which we solve, describes exactly how the changes occur. In fact, much of the reason that the calculus was developed by Isaac Newton was to describe motion and to solve differential equations. The bottom line is that many laws of nature relate the rate at which some quantity changes (the derivative) to the quantity itself.

Let's consider an example in classical mechanics. Suppose a particle of mass m moves along a line with constant velocity V_0. Suddenly, say at time $t = 0$, there is imposed an external resistive force F on the particle that is proportional to its velocity $v = v(t)$ for times $t > 0$. Intuitively, the particle will slow down and its velocity will change. From this information can we predict the velocity $v(t)$ of the particle at any time $t > 0$? We learned in calculus, and elementary physics, that Newton's second law of motion states that the mass of the particle times its acceleration equals the force upon it, or $ma = F$. We also learned that the derivative of velocity is acceleration, so $a = v'(t)$. Therefore, if we write the force as $F = -kv(t)$, where k is a proportionality constant and the minus sign indicates the force opposes the motion, then Newton's law implies

$$mv'(t) = -kv(t).$$

This is a differential equation for the unknown velocity $v(t)$. If we can find a function $v(t)$ that "works" in the equation, and also satisfies $v(0) = V_0$, then we will have determined the velocity of the particle at any time. Can you guess a solution? After some practice in Chapter 1 you will be able to solve the equation and find that the velocity decays exponentially; it is given by

$$v(t) = V_0 e^{-kt/m}, \ t \geq 0.$$

Let's check that it works:

$$mv'(t) = mV_0 \left(-\frac{k}{m}\right) e^{-kt/m} = -kV_0 e^{-kt/m} = -kv(t).$$

Moreover, substituting $t = 0$, we find $v(0) = V_0$. So it does check. The differential equation itself is a model that governs the dynamics of the particle. We set it up using Newton's second law, and it contains the unknown function $v(t)$, along with its derivative $v'(t)$. The solution $v(t)$ dictates how the system evolves in time.

Here is another example from demographics that shows how DEs arise naturally. Suppose the population of a small city is 100,000 people, and the population grows at a rate of 4% per year, while at the same time, there are 8000 emigrants out of the city each year. If $p = p(t)$ is the population at time t , what DE can we write down that describes how the population changes? Notice that the rate of change of the population is the derivative, or $p'(t)$. The statement of the problem tells us what the rates are: the growth rate is 4%, which states that the population increases by the amount $0.04p(t)$ each year; and the rate of emigration is a constant 8000 per year, which decreases the rate. So, we must have

$$p'(t) = 0.04p(t) - 8000,$$

which is a differential equation, or model, for the unknown population $p = p(t)$. The condition that there are initially 100,000 inhabitants can be translated into the mathematical condition that $p(0) = 100,000$, which is called an initial condition, and it puts a constraint on the possible solutions. This type of problem is typical in differential equations. Concisely, we are to solve

$$p' = 0.04p - 8000 \quad \text{subject to} \quad p(0) = 100,000.$$

As usual, we have not written the dependence of p on t; it is understood. Again we note that the DE relates an unknown function to its rate, which is typical in natural laws. The solution is

$$p(t) = 200,000 - 100,000e^{0.04t}.$$

Clearly, the population of the city is decreasing. (When is $p(t) = 0$?)

Historically, differential equations date to the mid-seventeenth century when the calculus was developed by Isaac Newton (c. 1665) in the context of determining the laws of mechanics (published in *Principia*, 1687). In fact, some would say that calculus was invented to describe how objects move. Afterwards, many of the who's who in mathematics and science, for example, L. Euler in the 1700s and A. Cauchy in the 1800s, developed the subject further and differential equations have become the principal tool in applications in all areas of mechanics, thermodynamics, electromagnetic theory, quantum theory, and so on. It continues today with the study of dynamical systems and nonlinear phenomena in biology, chemistry, economics, and almost every area where the dynamics of systems is important.

In this text we study differential equations and their applications. We mostly address two principal questions. (1) How do we find an appropriate DE model that describes a physical problem? (2) How do we understand or solve the DE after we obtain it? We learn modeling by examining models that others have studied (such as Newton's second law), and we try to create some of our own in the exercises. We gain understanding and learn solution techniques by practice.

Now we are ready. Read the text carefully with pencil and paper in hand, and work through all the examples. Make a commitment to solve most of the exercises. Keep in mind that DEs come from natural laws, many of which involve rates that processes occur. You will be rewarded with a knowledge of one of the monuments of mathematics and science, and you will see the great connection between nature and mathematics like you may never have imagined.