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# Nonlinear Differential Equations of Monotone Types in Banach Spaces

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# Preface

In the last decades, functional methods played an increasing role in the qualitative theory of partial differential equations. The spectral methods and theory of $C_0$-semigroups of linear operators as well as Leray–Schauder degree theory, fixed point theorems, and theory of maximal monotone nonlinear operators are now essential functional tools for the treatment of linear and nonlinear boundary value problems associated with partial differential equations.

An important step was the extension in the early seventies of the nonlinear dynamics of accretive (dissipative) type of the Hille–Yosida theory of $C_0$-semigroups of linear continuous operators. The main achievement was that the Cauchy problem associated with nonlinear $m$-accretive operators in Banach spaces is well posed and the corresponding dynamic is expressed by the Peano exponential formula from finite-dimensional theory. This fundamental result is the corner stone of the whole existence theory of nonlinear infinite dynamics of dissipative type and its contribution to the development of the modern theory of nonlinear partial differential equations cannot be underestimated.

Previously, in mid-sixties, some spectacular properties of maximal monotone operators and their close relationship with convex analysis and $m$-accretivity were revealed. In fact, Minty's discovery that in Hilbert spaces nonlinear maximal monotone operators coincide with $m$-accretive operators was crucial for the development of the theory. Although with respect to the middle and end of the seventies, little new material on this subject has come to light, the field of applications grew up and still remains in actuality. In the meantime, some excellent monographs were published where these topics were treated exhaustively and with extensive bibliographical references. Here, we confine ourselves to the presentation of basic results of the theory of nonlinear operators of monotone type and the corresponding dynamics generated in Banach spaces. These subjects were also treated in the author's books *Nonlinear Semigroups and Differential Equations in Banach Spaces* (Noordhoff, 1976) and *Analysis and Control of Nonlinear Infinite Dimensional Systems* (Academic Press, 1993), but the present book is more oriented to applications. We refrain from an exhaustive treatment or details that would divert us from the principal aim of this book: the presentation of essential results of the theory and its illustration by significant problems of nonlinear partial differential equations. Our aim is to present functional tools for the study of a large class of nonlinear problems and open to the reader the way towards applications. This book can serve as a teaching text for graduate students and it is self-contained. One assumes, however, basic knowledge of real and functional analysis as well as of differential equations. The literature on this argument is so vast and accessible in print that I have dispensed with detailed references or bibliographical comments. I have confined myself to a selected bibliography arranged at the end of each chapter.

The present book is based on a graduate course given by the author at the University of Iaşi during the past twenty years as well as on one-semester graduate courses at the University of Virginia in 2005 and the University of Trento in 2008.

In the preparation of the present book, I have received valuable help from my colleagues, Ioan Vrabie and Cătălin Lefter (Al.I. Cuza University of Iaşi), Gabriela Marinoschi (Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy) and Luca Lorenzi from University of Parma, who read the preliminary draft of the book and made numerous comments and suggestions which have permitted me to improve the presentation and correct the errors. Elena Mocanu from the Institute of Mathematics in Iaşi has done a great job in preparing and processing this text for printing.

Iaşi, September 2009 *Viorel Barbu*

# Acronyms

| | |
|---|---|
| $\mathbf{R}$ | the real line $(-\infty, \infty)$ |
| $\mathbf{R}^N$ | the $N$-dimensional Euclidean space |
| $\mathbf{R}^+ = (0, +\infty)$, | |
| $\mathbf{R}^- = (-\infty, 0)$, | |
| $\overline{\mathbf{R}} = (-\infty, +\infty]$, | |
| $\mathbf{R}^N_+ = \{(x_1, ..., x_N); x_N > 0\}$ | |
| $\Omega$ | open subset of $\mathbf{R}^N$ |
| $\partial\Omega$ | the boundary of $\Omega$ |
| $Q = \Omega \times (0, T)$, | |
| $\Sigma = \partial\Omega \times (0, T)$, | where $0 < T < \infty$ |
| $\\|\cdot\\|_X$ | the norm of a linear normed space $X$ |
| $X^*$ | the dual of space $X$ |
| $L(X, Y)$ | the space of linear continuous operators from $X$ to $Y$ |
| $\nabla f$ | the gradient of the map $f : X \to \mathbf{R}$ |
| $\partial f$ | the subdifferential of $f : X \to \mathbf{R}$ |
| $B^*$ | the adjoint of the operator $B$ |
| $\overline{C}$ | the closure of the set $C$ |
| $\operatorname{int} C$ | the interior of $C$ |
| $\operatorname{conv} C$ | the convex hull of $C$ |
| sign | the signum function on $X$ : $\operatorname{sign} x = x/\\|x\\|_X$ if $x \neq 0$ $\operatorname{sign} 0 = \{x;\ \\|x\\| \leq 1\}$ |
| $C^k(\Omega)$ | the space of real-valued functions on $\Omega$ that are continuously differentiable up to order $k$, $0 \leq k \leq \infty$ |
| $C^k_0(\Omega)$ | the subspace of functions in $C^k(\Omega)$ with compact support in $\Omega$ |
| $\mathscr{D}(\Omega)$ | the space $C^\infty_0(\Omega)$ |
| $\frac{d^k u}{dt^k}$, $u^{(k)}$ | the derivative of order $k$ of $u : [a, b] \to X$ |
| $\mathscr{D}'(\Omega)$ | the dual of $\mathscr{D}(\Omega)$ (i.e., the space of distributions on $\Omega$) |
| $C(\overline{\Omega})$ | the space of continuous functions on $\overline{\Omega}$ |

| | |
|---|---|
| $L^p(\Omega)$ | the space of $p$-summable functions $u : \Omega \to \mathbf{R}$ endowed with the norm $\|u\|_p = (\int_\Omega |u(x)|^p dx)^{1/p}$, $1 \le p < \infty$, $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$ for $p = \infty$ |
| $L^p_m(\Omega)$ | the space of $p$-summable functions $u : \Omega \to \mathbf{R}^m$ |
| $W^{m,p}(\Omega)$ | the Sobolev space $\{u \in L^p(\Omega); D^\alpha u \in L^p(\Omega), |\alpha| \le m, 1 \le p \le \infty\}$ |
| $W^{m,p}_0(\Omega)$ | the closure of $C^\infty_0(\Omega)$ in the norm of $W^{m,p}(\Omega)$ |
| $W^{-m,q}(\Omega)$ | the dual of $W^{m,p}_0(\Omega)$; $(1/p) + (1/q) = 1$, $p < \infty$, $q > 1$ |
| $H^k(\Omega), H^k_0(\Omega)$ | the spaces $W^{k,2}(\Omega)$ and $W^{k,2}_0(\Omega)$, respectively |
| $L^p(a,b;X)$ | the space of $p$-summable functions from $(a,b)$ to $X$ (Banach space) $1 \le p \le \infty$, $-\infty \le a < b \le \infty$ |
| $AC([a,b];X)$ | the space of absolutely continuous functions from $[a,b]$ to $X$ |
| $BV([a,b];X)$ | the space of functions with bounded variation on $[a,b]$ |
| $BV(\Omega)$ | the space of functions with bounded variation on $\Omega$ |
| $W^{1,p}([a,b];X)$ | the space $\left\{u \in AC([a,b];X);\ du/dt \in L^p([a,b];X)\right\}$ |