DARBOUX TRANSFORMATIONS IN INTEGRABLE SYSTEMS
MA THEM A TICAL PHYSICS STUDIES

Editorial Board:

Maxim Kontsevich, IHES, Bures-sur-Yvette, France
Massimo Porrati, New York University, New York, U.S.A.
Vladimir Matveev, Université Bourgogne, Dijon, France
Daniel Sternheimer, Université Bourgogne, Dijon, France
Darboux Transformations in Integrable Systems
Theory and their Applications to Geometry

by

Chaohao Gu
Fudan University, Shanghai, China

Hesheng Hu
Fudan University, Shanghai, China

and

Zixiang Zhou
Fudan University, Shanghai, China

Springer
## Contents

Preface ix

1. **1+1 DIMENSIONAL INTEGRABLE SYSTEMS** 1

   1.1 KdV equation, MKdV equation and their Darboux transformations 1
       1.1.1 Original Darboux transformation 1
       1.1.2 Darboux transformation for KdV equation 2
       1.1.3 Darboux transformation for MKdV equation 3
       1.1.4 Examples: single and double soliton solutions 6
       1.1.5 Relation between Darboux transformations for KdV equation and MKdV equation 10

   1.2 AKNS system 11
       1.2.1 $2 \times 2$ AKNS system 11
       1.2.2 $N \times N$ AKNS system 16

   1.3 Darboux transformation 18
       1.3.1 Darboux transformation for AKNS system 18
       1.3.2 Invariance of equations under Darboux transformations 23
       1.3.3 Darboux transformations of higher degree and the theorem of permutability 25
       1.3.4 More results on the Darboux matrices of degree one 30

   1.4 KdV hierarchy, MKdV-SG hierarchy, NLS hierarchy and AKNS system with $u(N)$ reduction 34
       1.4.1 KdV hierarchy 35
       1.4.2 MKdV-SG hierarchy 40
       1.4.3 NLS hierarchy 46
       1.4.4 AKNS system with $u(N)$ reduction 48
1.5 Darboux transformation and scattering, inverse scattering theory  
   1.5.1 Outline of the scattering and inverse scattering theory for the $2 \times 2$ AKNS system  
   1.5.2 Change of scattering data under Darboux transformations for $su(2)$ AKNS system

2. 2+1 DIMENSIONAL INTEGRABLE SYSTEMS  
   2.1 KP equation and its Darboux transformation  
   2.2 2+1 dimensional AKNS system and DS equation  
   2.3 Darboux transformation  
      2.3.1 General Lax pair  
      2.3.2 Darboux transformation of degree one  
      2.3.3 Darboux transformation of higher degree and the theorem of permutability  
   2.4 Darboux transformation and binary Darboux transformation for DS equation  
      2.4.1 Darboux transformation for DSII equation  
      2.4.2 Darboux transformation and binary Darboux transformation for DSI equation  
   2.5 Application to 1+1 dimensional Gelfand-Dickey system  
   2.6 Nonlinear constraints and Darboux transformation in 2+1 dimensions

3. $N + 1$ DIMENSIONAL INTEGRABLE SYSTEMS  
   3.1 $n + 1$ dimensional AKNS system  
      3.1.1 $n + 1$ dimensional AKNS system  
      3.1.2 Examples  
   3.2 Darboux transformation and soliton solutions  
      3.2.1 Darboux transformation  
      3.2.2 $u(N)$ case  
      3.2.3 Soliton solutions  
   3.3 A reduced system on $R^n$

4. SURFACES OF CONSTANT CURVATURE, BÄCKLUND CONGRUENCES  
   4.1 Theory of surfaces in the Euclidean space $R^3$  
   4.2 Surfaces of constant negative Gauss curvature, sine-Gordon equation and Bäcklund transformations
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2.1</td>
<td>Relation between sine-Gordon equation and surface of constant negative Gauss curvature in $\mathbb{R}^3$</td>
<td>126</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Pseudo-spherical congruence</td>
<td>129</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Bäcklund transformation</td>
<td>132</td>
</tr>
<tr>
<td>4.2.4</td>
<td>Darboux transformation</td>
<td>135</td>
</tr>
<tr>
<td>4.2.5</td>
<td>Example</td>
<td>139</td>
</tr>
<tr>
<td>4.3</td>
<td>Surface of constant Gauss curvature in the Minkowski space $\mathbb{R}^{2,1}$ and pseudo-spherical congruence</td>
<td>141</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Theory of surfaces in the Minkowski space $\mathbb{R}^{2,1}$</td>
<td>141</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Chebyshev coordinates for surfaces of constant Gauss curvature</td>
<td>144</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Pseudo-spherical congruence in $\mathbb{R}^{2,1}$</td>
<td>149</td>
</tr>
<tr>
<td>4.3.4</td>
<td>Bäcklund transformation and Darboux transformation for surfaces of constant Gauss curvature in $\mathbb{R}^{2,1}$</td>
<td>154</td>
</tr>
<tr>
<td>4.4</td>
<td>Orthogonal frame and Lax pair</td>
<td>174</td>
</tr>
<tr>
<td>4.5</td>
<td>Surface of constant mean curvature</td>
<td>179</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Parallel surface in Euclidean space</td>
<td>179</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Construction of surfaces</td>
<td>181</td>
</tr>
<tr>
<td>4.5.3</td>
<td>The case in Minkowski space</td>
<td>184</td>
</tr>
<tr>
<td>5.</td>
<td>DARBOUX TRANSFORMATION AND HARMONIC MAP</td>
<td>189</td>
</tr>
<tr>
<td>5.1</td>
<td>Definition of harmonic map and basic equations</td>
<td>189</td>
</tr>
<tr>
<td>5.2</td>
<td>Harmonic maps from $\mathbb{R}^2$ or $\mathbb{S}^{1,1}$ to $\mathbb{S}^2$, $H^2$ or $S^{1,1}$</td>
<td>192</td>
</tr>
<tr>
<td>5.3</td>
<td>Harmonic maps from $\mathbb{S}^{1,1}$ to $U(N)$</td>
<td>199</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Riemannian metric on $U(N)$</td>
<td>199</td>
</tr>
<tr>
<td>5.3.2</td>
<td>Harmonic maps from $\mathbb{S}^{1,1}$ to $U(N)$</td>
<td>201</td>
</tr>
<tr>
<td>5.3.3</td>
<td>Single soliton solutions</td>
<td>207</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Multi-soliton solutions</td>
<td>210</td>
</tr>
<tr>
<td>5.4</td>
<td>Harmonic maps from $\mathbb{R}^2$ to $U(N)$</td>
<td>213</td>
</tr>
<tr>
<td>5.4.1</td>
<td>Harmonic maps from $\mathbb{R}^2$ to $U(N)$ and their Darboux transformations</td>
<td>213</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Soliton solutions</td>
<td>219</td>
</tr>
<tr>
<td>5.4.3</td>
<td>Uniton</td>
<td>220</td>
</tr>
<tr>
<td>5.4.4</td>
<td>Darboux transformation and singular Darboux transformation for unitons</td>
<td>225</td>
</tr>
<tr>
<td>6.</td>
<td>GENERALIZED SELF-DUAL YANG-MILLS AND YANG-MILLS-HIGGS EQUATIONS</td>
<td>237</td>
</tr>
<tr>
<td>6.1</td>
<td>Generalized self-dual Yang-Mills flow</td>
<td>237</td>
</tr>
</tbody>
</table>
6.1.1 Generalized self-dual Yang-Mills flow 237
6.1.2 Darboux transformation 242
6.1.3 Example 245
6.1.4 Relation with AKNS system 247

6.2 Yang-Mills-Higgs field in 2+1 dimensional Minkowski space-time 248
6.2.1 Yang-Mills-Higgs field 248
6.2.2 Darboux Transformations 249
6.2.3 Soliton solutions 251

6.3 Yang-Mills-Higgs field in 2+1 dimensional anti-de Sitter space-time 256
6.3.1 Equations and their Lax pair 256
6.3.2 Darboux transformations 257
6.3.3 Soliton solutions in SU(2) case 260
6.3.4 Comparison with the solutions in Minkowski space-time 263

7. TWO DIMENSIONAL TODA EQUATIONS AND LAPLACE SEQUENCES OF SURFACES 267
7.1 Signed Toda equations 267
7.2 Laplace sequences of surfaces in projective space P_{n-1} 271
7.3 Darboux transformation 277
7.4 Su chain (Finikoff configuration) 281
7.5 Elliptic version of Laplace sequence of surfaces in CP^n 291
7.5.1 Laplace sequence in CP^n 291
7.5.2 Equations of harmonic maps from R^2 to CP^n in homogeneous coordinates 292
7.5.3 Cases of indefinite metric 295
7.5.4 Harmonic maps from R^{1,1} 296
7.5.5 Examples of harmonic sequences from R^2 to CP^n or R^{1,1} to CP^n 296

References 299
The soliton theory is an important branch of nonlinear science. On one hand, it describes various kinds of stable motions appearing in nature, such as solitary water wave, solitary signals in optical fibre etc., and has many applications in science and technology (like optical signal communication). On the other hand, it gives many effective methods of getting explicit solutions of nonlinear partial differential equations. Therefore, it has attracted much attention from physicists as well as mathematicians.

Nonlinear partial differential equations appear in many scientific problems. Getting explicit solutions is usually a difficult task. Only in certain special cases can the solutions be written down explicitly. However, for many soliton equations, people have found quite a few methods to get explicit solutions. The most famous ones are the inverse scattering method, Bäcklund transformation etc.. The inverse scattering method is based on the spectral theory of ordinary differential equations. The Cauchy problem of many soliton equations can be transformed to solving a system of linear integral equations. Explicit solutions can be derived when the kernel of the integral equation is degenerate. The Bäcklund transformation gives a new solution from a known solution by solving a system of completely integrable partial differential equations. Some complicated “nonlinear superposition formula” arise to substitute the superposition principle in linear science.

However, if the kernel of the integral equation is not degenerate, it is very difficult to get the explicit expressions of the solutions via the inverse scattering method. For the Bäcklund transformation, the nonlinear superposition formula is not easy to be obtained in general.
late 1970s, it was discovered by V. B. Matveev that a method given by G. Darboux a century ago for the spectral problem of second order ordinary differential equations can be extended to some important soliton equations. This method was called Darboux transformation. After that, it was found that this method is very effective for many partial differential equations. It is now playing an important role in mechanics, physics and differential geometry. V. B. Matveev and M. A. Salle published an important monograph [80] on this topic in 1991. Besides, an interesting monograph of C. Rogers and W. K. Schief [90] with many recent results was published in 2002.

The present monograph contains the Darboux transformations in matrix form and provides purely algebraic algorithms for constructing explicit solutions. Consequently, a basis of using symbolic calculations to obtain explicit exact solutions for many integrable systems is established. Moreover, the behavior of simple and multi-solutions, even in multi-dimensional cases, can be elucidated clearly. The method covers a series of important topics such as varies kinds of AKNS systems in $\mathbb{R}^{n+1}$, the construction of Bäcklund congruences and surfaces of constant Gauss curvature in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, harmonic maps from two dimensional manifolds to the Lie group $U(n)$, self-dual Yang-Mills fields and the generalizations to higher dimensional case, Yang-Mills-Higgs fields in $2 + 1$ dimensional Minkowski and anti-de Sitter space, Laplace sequences of surfaces in projective spaces and two dimensional Toda equations. All these cases are stated in details. In geometric problems, the Lax pair is not only a tool, but also a geometric object to be studied. Many results in this monograph are obtained by the authors in recent years.

This monograph is partially supported by the Chinese Major State Basic Research Program “Frontier problems in nonlinear sciences”, the Doctoral Program Foundation of the Ministry of Education of China, National Natural Science Foundation of China and Science Foundation of Shanghai Science Committee. Most work in this monograph was done in the Institute of Mathematics of Fudan University.