Young Measures on Topological Spaces

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Young Measures on Topological Spaces

With Applications in Control Theory and Probability Theory

by

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à Juliane, Marie-Blanche et Marie-Hélène

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Preface

Classical examples of more and more oscillating real–valued functions on a domain Ω of $\mathbb{R}^N$ are the functions $u_n(x) = \sin(nx_1)$ with $x = (x_1, \dots, x_n)$ or the so-called Rademacher functions on $]0,1[$, $u_n(x) = \mathfrak{r}_n(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x))$ (see later 3.1.4). They may appear as the gradients ∇v_n of minimizing sequences $(v_n)_{n\in\mathbb{N}}$ in some variational problems. In these examples, the function u_n converges in some sense to a measure μ on $\Omega \times \mathbb{R}$, called *Young measure.* In Functional Analysis formulation, this is the narrow convergence to μ of the image of the Lebesgue measure on Ω by $\omega \mapsto (\omega, u_n(\omega))$. In the disintegrated form $(\mu_\omega)_{\omega\in\Omega}$, the *parametrized measure* μ_ω captures the possible scattering of the u_n around ω.

Curiously if $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables deriving from independent ones, the n-th one may appear more and more far from the k first ones as if it was oscillating (think of orthonormal vectors in L^2 which converge weakly to 0). More precisely when the laws $\mathrm{L}(X_n)$ narrowly converge to some probability measure ϖ, it often happens that for any k and any A in the algebra generated by $X_1, \dots, X_k$, the conditional law $\mathrm{L}(X_n|A)$ still converges to ϖ (see Chapter 9) which means

$$\forall \varphi \in \mathrm{C}_b(\mathbb{R}) \quad \frac{1}{\mathrm{P}(A)} \int_A \varphi(X_n(\omega))\, d\mathrm{P}(\omega) \longrightarrow \int_{\mathbb{R}} \varphi \, d\varpi$$

or equivalently, $\underline{\delta}_{X_n}$ denoting the image of P by $\omega \mapsto (\omega, X_n(\omega))$,

$$\int_{\Omega\times\mathbb{R}} (\mathbb{1}_A \otimes \varphi)\, d\underline{\delta}_{X_n} \longrightarrow \int_{\Omega\times\mathbb{R}} (\mathbb{1}_A \otimes \varphi)\, d[\mathrm{P} \otimes \varpi]\,.$$

This is exactly the same convergence as the one raised in the first paragraph (excepted that the limit measure is not always a product).

Many authors wrote on Young measures in Control and Calculus of Variations: Young [You37], McShane [McS40], Gamkrelidze [Gam62, Gam78], Warga [War72], Ghouila-Houri [GH67], Tartar [Tar78], Ekeland [Eke72], Berliocchi and Lasry [BL71, BL73], Balder in [Bal95, Bal00a, Bal00b] and many other papers cited in this book. On the probabilistic side, we refer to Rényi [Rén58, Rén66, Rén63] for applications in limit theorems, Baxter and Chacon [BC77] and Meyer [Mey78] for relaxed stopping times, Pellaumail for weak solutions of stochastic differential equations [Pel80, Pel81] (see also Jacod and Mémin [JM81b, JM81a]).

The topology on the space of Young measures is usually called “narrow topology” or “weak topology” in the study of Young measures in Functional Analysis. But this topology is called “stable” in Probability Theory, and the word “stable” has the advantage that it avoids confusions with the usual narrow topology on the space of measures on a topological space (see more details on this discussion page 22). So, we choose here “stable topology”.

For a long time Young measures were considered only when the functions u_n take their values in a compact subset of an Euclidean space. Berliocchi and Lasry introduced locally compact spaces and Balder extended the Prohorov theorem to these parametrized measures that are Young measures. Then during a long period authors considered that the good space was a Polish or a metrizable Suslin space. Between 1985 and 1990 several works (mainly due to Balder) treated the case of a separable reflexive Banach with the weak topology which is not metrizable. For an example of Young measures on a function space, see [MV02]. In this book the extension of the compactness Topsøe criterion to Young measures allows a significant progress. We will consider basically (but always adding some technical topological hypotheses) a general Hausdorff topological space.

Contents Our aim is not to give a short introduction to Young measures. Instead, we present the results in a general setting, in the hope it will be useful for further developments of the theory. Due to the general framework four stable topologies are introduced in Chapter 2. Dudley's results are used to study in Chapter 3 convergence in probability of the functions $\omega \mapsto \mu^\alpha_\omega$ where α is the index of a net and $(\mu^\alpha_\omega)_{\omega\in\Omega}$ denotes the disintegration of μ^α (which is assumed to exist). A general fiber product theorem and a parametrized Kantorovich-Rubinštein theorem are provided. The heart is Chapter 4 where the Topsøe criterion is extended to Young measures. Chapter 6 is devoted to vector valued functions, the biting Lemma, weak compactness results in $L^1_{\mathbb{E}}$ and Visintin's theorem in several infinite dimensional frameworks. Chapter 7 develops several relaxation results in Control and evolution problems. Chapter 8 gives some results of Calculus of Variations: the lower semicontinuity theorem and Reshetnyak's theorem, and deals with the fiber product of Young measures and its applications to control problems: essentially we establish the link between the value function which occurs in these problems and the viscosity solution of the associated Hamilton–Jacobi–Bellman equation. Finally Chapter 9 gives some results from Probability Theory which involve stable convergence.

There are many directions that we did not investigate... Specially the result of Kinderlehrer and Pedregal about Young measures generated by gradients of vector valued functions (as this necessarily happens in some physical problems where the functions describe the deformation of a 3-dimensional material): see the books of Roubíček [Rou97] and Pedregal [Ped97], see also [Syc98, Syc99] and the forthcoming book [ABM]. We did not either investigate some generalizations of Young measures which should be very useful in an infinite dimensional setting, especially in the case of nonseparable spaces; for example, we did not consider Young measures with cylindrical values in a Banach space, nor Young measures which have only finitely additive values (such measures are considered by Fattorini [Fat99]).

Definitions are spread all along the text, without any numbering. The reader should consult the Subject Index and the Index of Notations to find their precise location.

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