Plato's Problem

Also by Marco Panza

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Plato's Problem

An Introduction to Mathematical Platonism

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Preface

The aim of this book, a revised and partially enlarged edition of the Italian original *Il Problema di Platone* published in 2010 (Carocci Editore, Roma), is to offer an introduction to the philosophy of mathematics addressed to all those wishing to familiarize themselves with the subject, and even to those lacking any previous acquaintance with it.

We have done our best to organize the discussion so as to avoid reliance on previous knowledge. An introduction, however, cannot cover everything, and the philosophy of mathematics is a vast domain with a proliferation of close connections to several other domains: mathematics itself and its history, logic, philosophy of language, history of philosophy, just to take some obvious examples. The reader will thus unavoidably encounter remarks for which full understanding requires acquaintance, if only at basic levels, with different disciplines.

Several other introductions to the philosophy of mathematics are available. Mentioning a few of them, and limiting ourselves to books in English, we can point to Brown (1999), Potter (2002), Shapiro (2000a), George and Velleman (2001), Giaquinto (2002), Mancosu (2008a), Bostock (2009), Colyvan (2012). Our book will obviously discuss many topics that are covered also in these introductions, but it is intended to approach them with a particular focus.

Introductions to the philosophy of mathematics have often aimed at offering a comprehensive account of the discussion in the field since at least the end of the nineteenth century. In some cases, the historical background is kept to a minimum, in order to leave room for a theoretical discussion of the main philosophical options available in the contemporary debate. We have followed a different strategy. We have focused on a single problem and a single option that has been offered as a solution of it; we then trace the main historical development of this latter option (Chapters 1–2), discuss the debate it has engendered from the 1960s to the present day (Chapters 3–5) and finally focus in more detail on a currently widely discussed argument that has been advanced in that debate (Chapters 6–7).

The problem we have focused on is that of the ontology of mathematics, the problem that in our title we call 'Plato's problem' (the reasons for this will be clear in § 1.1). The following is a very simple way of presenting it: granted that the statements of mathematics are about something, what are they about? The answer we focus on goes under the name of 'platonism'. It claims, broadly speaking, that these statements are about a domain of abstract objects, which they describe.

We are well aware that this way of proceeding can only offer a very partial representation of past and present philosophy of mathematics, which has dealt with several other problems and answers. Rather than presenting a broad-brush summary of this vast range of topics, we prefer to limit the scope of our introduction so as to make it possible to go into much more detail on the option being discussed and the specific argument in its support. Chapters 6 and 7 are devoted to this latter, the so-called 'indispensability argument'. Our hope is that the reader will be guided through a significant part of present-day discussion in the philosophy of mathematics, possibly paving the way for further studies. In order to achieve this, the book's chapters have been written in three different styles.

Chapters 1 and 2 contain an (inevitably partial) historical reconstruction, with authors and topics considered in chronological order. Connections among specific topics are emphasized. The aim of these chapters is to offer the historical background that seems to us required in order to understand why the contemporary debate is so relevant. Philosophy, even in its most technical and specific regions, is largely motivated by its history and tradition, and no philosophical discussion can really be appreciated when these are wholly disregarded.

Chapters 3, 4 and 5 offer a synchronic reconstruction of several options, some supporting platonism and some opposing it. We have chosen to arrange them depending on which sort of response they offer to the dilemma advanced by Paul Benacerraf in the early 1970s, a dilemma that in our opinion can easily be seen as a modern version of Plato's problem. The aim of these chapters is to present an articulated summary of the current discussion on this problem, one that the philosophy of mathematics, as the previous chapters show, inherits from its tradition.

Chapters 6 and 7 present a systematic reconstruction of a particular argument, aiming to show its assumptions, motivations and difficulties, together with the necessary and/or sufficient conditions for its conclusions. We have opted for a detailed, and often explorative, analysis of the various theoretical ingredients involved in one single argument, an argument that has been suggested during the course of the past fifty years by one of the most authoritative thinkers in the empiricist tradition, Willard van Orman Quine. The contrast between this tradition and the platonic one, that finds a point of intersection in the indispensability argument, is one of its most interesting traits.

We have chosen to devote two chapters to this argument since it involves several crucial issues in contemporary philosophy of mathematics. This is not to say that we intend to appeal to the argument (once appropriately stated) in order to convince the reader of the correctness of the platonist option in the version it supports. Quite the contrary, we believe this argument to have various limitations and difficulties, many of which will be discussed. Our only purpose is to familiarize the reader with the intricacies of contemporary philosophy of mathematics through the consideration of a specific example among the current debate.

Even the choice of treating Plato's problem is partial. Other problems differ from it not only as regards their content, but also as regards their nature. Roughly speaking, we can single out four kinds of problems pertaining to the philosophy of mathematics.

First, there are foundational problems concerning the best way of founding, justifying and organizing the edifice of mathematics or at least some relevant parts of it. Next, there are general interpretative issues relating to the nature of mathematics itself, such as Plato's problem and other problems variously related to it, such as the problem of the nature of mathematical knowledge. (Is there mathematical knowledge, and if so, what kind of knowledge is it?) or that of the logical character of the truths or theorems of mathematics. (If there are any truths in mathematics, what is their source? Are they analytic or synthetic? A priori or empirical? And if there are none, what legitimates the theorems of mathematics? Just to give an example, if '3 + 5 = 8' is not a true statement, in what way does it differ from '3 + 5 = 9'?)

There are also more specific interpretive problems, concerned with particular mathematical theories, or with mathematical practice as it has developed over time. Mancosu (2008a) offers an excellent survey of some of these problems, such as that of the availability of a criterion for acknowledging when arguments are explanatory, or that of visualization, or more generally of diagrammatic reasoning in mathematics, or that of the possibility of offering a principle of purity, selecting some proofs or theories as being better than others.

Last, there are problems relating to the applicability of mathematics, the justification for this applicability and how it takes place, to the role mathematics has in empirical sciences and in our everyday lives.

Many philosophers of mathematics believed, and still believe, that their main task is to celebrate the beauty of mathematics. Carl Jacobi – a great mathematician of the first half of the nineteenth century maintained – against Joseph Fourier – that the only aim of mathematics is “the honour of human mind”. Even though we wholeheartedly acknowledge this greatness, and feel the fascination that many mathematical theories exert on us, we believe that this is not the main purpose of the philosophy of mathematics.

We are, then, left with the four sorts of problems mentioned above. They are not certainly unrelated, and indeed many of them, even when they fall under different categories, are so closely intertwined that it is hard to understand how one can be treated in isolation from the others. One of the characteristic traits of the way of pursuing philosophy that is usually labelled ‘analytic’, however, is that different problems are singled out and treated in their specific aspects in order to deal with them with the required precision, thus avoiding getting lost in overly general considerations. Many of the thinkers we discuss can surely be counted among the practitioners of this kind of philosophy. We ourselves strived to make the subtler distinctions possible, at least so far as the level of generality of an introduction allows. Still, the present book need not necessarily be viewed as an introduction to the analytic philosophy of mathematics.

This is mainly due to the fact that, contrary to what happens in several areas of contemporary philosophy, within the philosophy of mathematics the distinction between the analytic and rival approaches is not that sharp. One reason for this is that the philosophy of mathematics is necessarily receptive to the characters of mathematics itself and to the methodological worries of mathematicians, and this forces some lines of inquiry and naturally excludes others, independently of one’s preferred philosophical attitude. Another reason is that mathematics constantly undergoes technical evolutions, and is characterized by an historical development that cannot be openly denied without leading to utterly implausible views. This rules out any approach that might excessively abstract from actual mathematical practice and from historical considerations. A third reason is that the scenery of the philosophy of mathematics has been dominated, between the last decades of the nineteenth century and the early 1970s, by the problem of foundations, and this has inevitably led to a profound confrontation with authors such as Gottlob Frege, Bertrand Russell and Rudolf Carnap, who are unanimously seen as the fathers of the analytic tradition, and others like Henri Poincaré, David Hilbert, Luitzen Egbertus Jan Brouwer, Hermann Weyl and Kurt Gödel, who could hardly be seen as part of that tradition.

These are, in the end, only three different ways of articulating the very same reason, which has resulted in a varied mixture of sensibilities, styles, skills, formative experiences and points of view. For instance, problems falling under the third category described above are now being studied by philosophers that would hardly qualify themselves as analytic, and who nonetheless confront them with methods and assumptions that typically pertain to the analytic tradition. This does not prevent them from grounding many of their arguments on historical considerations or investigations on the technical details of mathematical theories. The philosophy of mathematics thus appears as a privileged field where, contrary to what happens in other areas of philosophy, collaborations and liaisons among different approaches can be the subject of fruitful experimentation.

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Terminological Conventions

In what follows, we lay down some terminological conventions that are adopted throughout the book. Like all conventions, they are partly arbitrary. They are often required for ease of exposition, and are not meant to rule out different uses that may be equally legitimate. Apart from those listed here, other conventions are adopted where needed, for which separate treatment would have been superfluous, since our use of the relevant expressions will make their intended meaning clear enough (throughout the book we have tried to make these uses as stable as possible). More specific conventions on less frequent words are introduced at specific points in the text.

By 'sentence (of a certain language)' we mean a formula (of that language) as such, as a mere succession of symbols or words subject to certain composition rules. A sentence is often considered for the function it plays (possibly on the basis of an appropriate interpretation of its symbols or words). If this function is to assert or to state something, namely that things stand in a particular way, then we say that it expresses a statement. This statement will then be said to be formulated in that sentence's language. If an appropriate equivalence relation is defined on the sentences of a certain language, the same statement will be expressible by different sentences of that language, each of which will be said to consist in a different formulation of it. Once one of these formulations is given, others are often referred to as reformulations of the statement it expresses. The equivalence relation between sentences expressing the same statements might sometimes be defined also over statements in different languages. We will then say that a given statement, as expressed by a sentence of one of these languages, has a reformulation or can be reformulated in other languages. A statement can thus be understood as an equivalence class of sentences of the same language or of different languages. It is clear, however, that it can only be displayed by one or another among these sentences. In brief, when a sentence is taken as a representative of a certain equivalence class of statements, it will then straightforwardly be called 'statement'.

We distinguish a statement, as a linguistic object, from the linguistic act of asserting something (this act being generally called 'assertion'). We also distinguish a statement from what it states, i.e. its content, which is generally called 'proposition'.

We take the expression ‘to make a statement’ to have two different senses. In a first sense, which we might call ‘propositional’, this expression means the act of stating that something is the case, that things stand as the statement (appropriately understood) says they stand. In a second sense, it means the act of performing an utterance with assertoric force: in this sense, ‘to make the statement *x*’ means the same as ‘to write φ’ or ‘to utter φ’, where ‘φ’ denotes the sentence expressing *x*. The context will make clear which of the two senses is relevant on each occasion.

In our book, we shall mainly deal with statements (we will consider sentences and propositions on rare occasions). This is due to the fact that, both in mathematics and in the empirical sciences, sentences are not usually studied as such – contrary to what happens in certain branches of logic or linguistics – but are rather considered for the statements they express.

We will use the term ‘theory’ to refer to a system of statements that we take, in most cases, to be or have been accepted by someone, generally by a whole scientific community (this simply means that members of this community make or have made these statements or have manifested their proneness to do it). In order to stress the relations that are supposed to hold among these statements and/or in order to be faithful to the linguistic usage of authors, we will sometime speak of bodies of statements – though they will always be taken to form more or less structured theories. Generally, though not necessarily, we will take a theory to be a system of statements deductively closed under appropriate deductive rules (which means that if the theory includes certain statements it also includes all those – generally infinitely many – statements that follow from the former by these rules). Sometimes we will speak of parts or portions of a theory, or of sub-theories, meaning by this to refer to appropriately selected sub-systems of the system of statements forming a given theory (these sub-systems being themselves theories). We will call ‘mathematical’ those theories that are usually taken to be such, without specifying in general what this adjective exactly means. In accordance with widespread usage, we will on the other hand call ‘scientific’ those theories usually pertaining to empirical sciences, as distinct not only from mathematical ones, but also from logical, metaphysical, etc.

We will speak of statements of mathematics, of arithmetic, or of geometry in order to refer, respectively, to statements belonging to a given mathematical theory, to a given version of arithmetic, or to a given geometrical theory. These statements are sometimes called ‘theorems’ or 'consequences (of the relevant theory)'. This holds especially when it is important to stress that these statements follow – according to accepted inference rules – from other statements that will then be called 'axioms'. This happens usually for mathematical and logical theories, which are usually deductively closed. Sometimes we will distinguish pure from impure mathematical theories: the former are meant to include only statements that are formulated in a specifically mathematical language; the latter, on the contrary, are meant to include statements in which, together with the expressions of such a specifically mathematical language, expressions of a non-mathematical, generally scientific, language also occur. It thus follows that an impure mathematical theory can, and usually is, meant to be a scientific theory, even though obviously not all scientific theories are impure mathematical theories.

By 'mathematical statement' we mean, more generally, a statement that is formulated in the language of a given mathematical theory (usually a pure one, but possibly also an impure one, so long as it features expressions of a specifically mathematical language; in these cases we will also speak of 'impure mathematical statements'). By 'nominalistic statement' we mean a statement formulated in a purely nominalistic language (in § 4.1 we will explain what usually counts as a nominalistic language; approximately, it is a language which provides means for speaking only of concrete objects). When a mathematical statement is reformulated in a nominalistic language, we will speak of its nominalistic reformulation. Such a reformulation is often called 'paraphrase'. However, we will speak of paraphrases more generally also to refer to other reformulations of certain statements, not necessarily in a nominalistic language.

According to the foregoing conventions, '5 + 8 = 13' is both a mathematical statement (arithmetical, more precisely) and a statement of mathematics (of arithmetic, more precisely, or of any version of it), whereas the statement '5 + 8 = 12' is a mathematical (arithmetical) statement but is not a statement of mathematics (of arithmetic). The statement 'The acceleration of gravity at terrestrial poles is equal to 9,823 m/s^2' is a statement of an impure mathematical theory which is also a scientific theory, more precisely a version of the gravitational theory. The statement 'Earth poles shift in time' is a non-mathematical statement (save if one wishes to consider 'poles' as part of a geometrical language) belonging to a scientific theory, namely geology. Like statements such as 'Earth poles are always iced' or 'Earth poles are always flowery', it is also a nominalistic statement.

A theory will be said to be true if and only if all of its statements are true (according to a specified conception of truth).

We will use single inverted commas for mention (thus we will say that '3' is a singular term) and double inverted commas for quotation. We will use corners for expressions designating properties, relations or concepts: we will say for example that under the concept ⌜(being) identical to 1⌝ falls only one object.