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# Non-commutative Gelfand Theories

## A Tool-kit for Operator Theorists and Numerical Analysts

Springer

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*Dedicated to the memory of Israel Gohberg (1928–2009).*

# Preface

The central notion in this book is that of a local principle. Local principles provide an abstract frame for the natural and extremely useful idea of localization, i.e. to divide a global problem into a family of local problems. The local principles the reader will encounter in this text are formulated in the language of Banach algebras and can be characterized as non-commutative Gelfand theories. They now form an integral part of the theory of Banach and $C^*$-algebras, and they provide an indispensable tool to study concrete problems in operator theory and numerical analysis.

More than thirty years ago, Douglas derived the Gohberg-Krupnik symbol calculus for singular integral operators via a combination of a local principle (which now bears his name) and Halmos' two projections theorem. Around the same time Kozak proved the equivalence between the stability of an operator sequence and the invertibility of a related element in a certain Banach algebra which he then studied by Simonenko's local principle. Since that time there have appeared dozens of papers where the idea of localization has been used, further developed, and applied in several contexts. As the outcome of this development, we now have a powerful, rich and beautiful theory of algebraic localization, the principles and results of which are widely scattered in the literature. The lack of a general context, and the use of different notation from paper to paper make it difficult for the researcher and the graduate student to familiarize themselves with the theory behind local principles and to make use of these results to study their own problems.

It is this defect that the present book seeks to solve. It started as a much simpler task: an updated re-edition of an out-of-print report [168], back in 1998. The changing objectives and professional obligations of the authors kept on increasing the scope and delaying the work. After more than ten years, we are finally able to present it. We think that the delay has been worth it, and the reader has a readable and useful text in his hands.

It is our intention that this work be a basic but complete introduction to local principles, formulated in the language of Banach algebras, that allows the reader to get a general view of the area and enables him to read more specialized works. Many results that appeared in periodicals or reports, and can be hard to find, are presented, streamlined and contextualized here, and the relations with other results are made clear. Some results which were in complete form available only in the Russian literature, like the local principle by Simonenko, are included. And finally, a few existing gaps in the theory are filled in with full proofs, which appear here for the first time.

The text starts with a chapter on the relevant notions for local principles of Banach algebra theory. As such, the first part can serve as a textbook for a one semester graduate course on Banach algebras with emphasis on local principles. Exercises and examples are given throughout the text. We focus on applications to singular integral operators and convolution type operators on weighted Lebesgue spaces. The choice of applications is the result both of our particular interests as researchers and of the genetic inheritance of the text, which was born as a report on algebras of convolution type operators.

Most figures in the book were produced with the help of *Mathematica*[1]. A couple of figures were produced with *Adobe Illustrator*[2]. The authors acknowledge the research center CMA and its successor CEAF (Portugal) for travel and meeting support during the writing of the book.

We would like to thank our colleagues Marko Lindner and Helena Mascarenhas for their stimulating and helpful discussions during the work on this text. We are specially grateful to Alexei Karlovich, who carefully read the manuscript and gave many valuable suggestions for its improvement.

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[2] Adobe Illustrator is a trademark of Adobe, Inc.

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## Part II Case Studies

# Introduction

This is a text on tools which can help to solve invertibility problems in Banach algebras. The number of such problems is much larger than one might guess at first glance. Of course, the most obvious invertibility problem one has in mind is the question of whether a given operator is invertible in an algebra of operators, or whether a given function is invertible in an algebra of functions. A classical example, solved by Wiener, is the question of whether the inverse of a non-vanishing function in the algebra of functions with absolutely convergent Fourier series belongs to the same algebra again.

But there are many more problems in analysis, operator theory, or numerical analysis which turn out to be equivalent to invertibility problems in suitably associated Banach algebras. For example, think of the question of whether a given bounded linear operator on a Banach space possesses the Fredholm property, i.e., whether its kernel and its cokernel are linear spaces of finite dimension. One of the equivalent characterizations of this notion states that an operator has the Fredholm property if and only if its image in the Calkin algebra is invertible. Hence, Fredholmness is indeed an invertibility problem.

For technical reasons, this invertibility problem is often studied in a suitable subalgebra of the Calkin algebra. An example which will be treated in detail in this text is the smallest closed subalgebra of the Banach algebra of all bounded linear operators on $L^2(\mathbb{T})$ which contains all singular integral operators with continuous coefficients. Here, $\mathbb{T}$ denotes the complex unit circle, and $L^2(\mathbb{T})$ is the related Lebesgue space with respect to the normalized Lebesgue measure. The Calkin image of this subalgebra is a commutative $C^*$-algebra, and hence subject to the Gelfand-Naimark theorem. In particular, this algebra proves to be isometrically isomorphic to the algebra $C(\mathbb{T}) \times C(\mathbb{T})$, where $C(\mathbb{T})$ is the algebra of all complex-valued continuous functions on $\mathbb{T}$. In that sense, we know all about this algebra. However, if the coefficients are merely piecewise continuous, this subalgebra of the Calkin algebra is no longer commutative, and the classical Gelfand theory fails.

A different collection of examples stems from a problem in numerical analysis. To solve an operator equation $Au = v$ numerically, one chooses a sequence of operators $A_n$ which act on finite-dimensional spaces, and which converges strongly to $A$, and one replaces the equation $Au = v$ by the sequence of finite linear systems $A_n u_n = v_n$ with suitable approximations $v_n$ of the right-hand side $v$. A crucial question is whether the sequence $(A_n)$ is stable, i.e., whether the operators $A_n$ are invertible for large $n$ and whether the norms of their inverses are uniformly bounded.

The stability of a sequence of operators is again equivalent to an invertibility problem. For simplicity, assume that each $A_n$ is an $n \times n$ matrix. We consider the direct product $\mathscr{E}$ of the sequence $(\mathbb{C}^{n\times n})_{n\geq 1}$ of algebras, i.e., the set of all bounded sequences $(A_n)_{n\geq 1}$ of matrices $A_n \in \mathbb{C}^{n\times n}$. Provided with pointwise defined operations and the supremum norm, this set becomes a Banach algebra. Further, we write $\mathscr{G}$ for the restricted product of the sequence $(\mathbb{C}^{n\times n})_{n\geq 1}$, i.e., for the collection of all sequences $(G_n)_{n\geq 1} \in \mathscr{E}$ such that $\|G_n\| \to 0$ as $n \to \infty$. The set $\mathscr{G}$ is a closed two-sided ideal of $\mathscr{E}$. Now a simple Neumann series argument shows that a sequence $(A_n) \in \mathscr{E}$ is indeed stable if and only if its coset $(A_n) + \mathscr{G}$ is invertible in the quotient algebra $\mathscr{E}/\mathscr{G}$. Hence, stability is also an invertibility problem.

Commutative Banach algebras are subject to the Gelfand theory, one of the most beautiful pieces of functional analysis. The essence of this theory is given by the following observation: To each unital commutative Banach algebra $\mathscr{A}$, there is associated a compact Hausdorff space $M_{\mathscr{A}}$ such that $\mathscr{A}$ can be represented (up to elements in the radical) as an algebra $\widehat{\mathscr{A}}$ of continuous functions on $M_{\mathscr{A}}$. More precisely, there is a continuous homomorphism $\widehat{\ }: \mathscr{A} \to \widehat{\mathscr{A}}$ (called the Gelfand transform) which has the radical of $\mathscr{A}$ as its kernel, and which owns the following property: an element $a$ is invertible in $\mathscr{A}$ if and only if its Gelfand transform $\widehat{a}$ does not vanish on $M_{\mathscr{A}}$.

To state the latter fact in a different way note that, for each $x \in M_{\mathscr{A}}$, the point evaluation $a \mapsto \widehat{a}(x)$ defines a homomorphism from $\mathscr{A}$ onto $\mathbb{C}$, and one can show that every non-trivial homomorphism from $\mathscr{A}$ onto $\mathbb{C}$ is of this form. Thus, an element $a$ of a unital commutative Banach algebra $\mathscr{A}$ is invertible if and only if $\varphi(a) \neq 0$ for every non-trivial homomorphism $\varphi : \mathscr{A} \to \mathbb{C}$.

For general (non-commutative) unital Banach algebras $\mathscr{A}$, non-trivial homomorphisms from $\mathscr{A}$ onto $\mathbb{C}$ need not exist. Think of the Banach algebra $\mathbb{C}^{n\times n}$ with $n > 1$, which does not possess non-trivial ideals. To derive a theory for non-commutative Banach algebras which can serve as a substitute for the classical Gelfand theory it is therefore necessary to allow for more general homomorphisms on $\mathscr{A}$ rather than homomorphisms into $\mathbb{C}$. We shall see that such generalizations of the Gelfand theory indeed exist, provided the underlying algebra is not too far from a commutative algebra. In particular, we shall consider two classes of Banach algebras which satisfy this assumption: algebras which possess a rich center, and algebras which fulfill a standard polynomial identity. The center of an algebra consists of all elements which commute with each other element of the algebra. Thus, commutative algebras are algebras which coincide with their center, and algebras with a large center can thus be considered as close to commutative algebras. On the other hand, polynomial identities serve as a substitute for the simplest polynomial identity $ab = ba$, which characterizes the commutative algebras. In that sense also, algebras with polynomial identity are close to commutative algebras.

Algebras with a large center occur in many places. So it is no wonder that concepts for their study have been worked out since the nineteen-sixties. These concepts were called local principles, because the underlying ideas resemble the method of *localization* or *freezing of coefficients* widely used in the theory of partial differential equations. Local principles can indeed be considered as non-commutative Gelfand theories in the sense that they associate to a given Banach algebra $\mathscr{A}$ with a non-trivial central subalgebra $\mathscr{C}$, a family of Banach algebras $\mathscr{A}_\tau$ with continuous homomorphisms $\mathsf{W}_\tau : \mathscr{A} \to \mathscr{A}_\tau$ – labeled by the maximal ideals of $\mathscr{C}$ – such that an element $a \in \mathscr{A}$ is invertible in $\mathscr{A}$ if and only if its "shadow" $\mathsf{W}_\tau(a)$ is invertible in $\mathscr{A}_\tau$ for every $\tau$. Of course, one should expect that the invertibility of the elements $\mathsf{W}_\tau(a)$ is easier to verify than the invertibility of $a$ itself, in which case the local principle provides an effective tool to study invertibility.

The book starts with a concise exposition about Banach algebra theory centered around the notions of invertibility and spectrum. In Chapter 2, we study several local principles, namely the local principles by Allan-Douglas, Simonenko, and Gohberg-Krupnik. In their original form, they appeared about 40 years ago, and the relationship between them was not fully understood. The latter changed in the last years, mainly thanks to the introduction of new technical ingredients like norm-preserving localization, local inclusion theorems, and theorems of Weierstrass type.

Chapter 2 is concluded by a discussion of Krupnik's generalization of Gelfand theory to Banach algebras that fulfill a standard polynomial identity, the so-called PI-algebras. The latter proved to be extremely useful to study Banach algebras generated by idempotents (with some relations between them), which is the subject of Chapter 3. Our goal is to present this material, which until now has been spread over many publications, in a systematic way. These first three chapters form the first part of the book.

The second part of this text deals with case studies where local principles are applied to various particular Banach algebras generated by bounded operators of a special type or generated by approximation sequences of special operator classes. For instance, we shall consider algebras generated by one-dimensional singular integral operators with piecewise continuous coefficients on composed curves acting on $L^p$-spaces with Khvedelidze weights, and algebras generated by Wiener-Hopf and Hankel operators with piecewise continuous generating functions. The local principles will be employed to derive criteria for the Fredholm property of operators in these algebras. However we will not deal with index computation since it is not a matter of local theories but is of a global nature.

Among the concrete examples of algebras of approximation sequences we shall be concerned with in Chapter 6 are algebras of the finite sections method and of spline Galerkin methods for one- and two-dimensional singular integral and Wiener-Hopf operators. A peculiarity of these algebras is that their center is trivial in many cases. It is therefore a further goal of Chapter 6 to introduce some tools, the so-called lifting theorems, which allow one to overcome these difficulties by passing to a suitable quotient algebra which then has a nice center.

The authors have tried to make this book as easy to read as possible, giving special attention to coherence of notation throughout the book. Usually the font of a symbol will give an immediate clue to the type of mathematical object it represents. For example, sets (like those in the complex plane) are usually represented by the same font as the one used for the real line or the complex plane ($\mathbb{R}$, $\mathbb{C}$), while general curves in the complex plane are represented by $\Gamma$. Algebras and ideals are usually represented in a calligraphic font, as in $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, etc. Lower case letters $a$, $b$, $c$ can either represent elements of an abstract algebra (with $e$ the identity), or functions. In the case of functions we reserved $f$ and $g$ for continuous functions, $u$ and $v$ for elements of Lebesgue spaces, but $i-n$ we left for indexes. The imaginary unit is represented by $\mathbf{i}$. Upper case roman letters $A$, $B$, $C$ etc, usually represent operators, whereas $\mathsf{H}$ and $\mathsf{W}$ (specified by additional parameters) are used to designate homomorphisms.