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Applied and Numerical Harmonic Analysis

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Volker Michel

Lectures on Constructive Approximation

Fourier, Spline, and Wavelet Methods on the Real Line, the Sphere, and the Ball

Birkhäuser

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ISBN 978-0-8176-8402-0 ISBN 978-0-8176-8403-7 (eBook)
DOI 10.1007/978-0-8176-8403-7
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012948340

Mathematics Subject Classification (2010): 33C45, 33C50, 33C55, 33F05, 41-01, 41A05, 41A10, 41A15, 41A30, 41A50, 41A52, 41A63, 42A10, 42A15, 42A82, 42A85, 42C05, 42C10, 42C40, 43A90, 65D05, 65D07, 65D20, 65D30, 65D32, 65R32, 65T40, 65T50, 65T60, 86-08, 86A22

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.birkhauser-science.com)

This book is dedicated to the Father of Geomathematics,

Prof. Dr. Willi Freeden,

a passionate teacher, a visionary scientist, and a wise mentor, in acknowledgement of his infinite support and in the hope for many further joint projects to come.

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and time-scale analysis*
Numerical partial differential equations	*Wavelet theory*

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function". Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, e.g., by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but it also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunctionwith the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD

John J. Benedetto
Series Editor

Preface

This book is the result of numerous courses titled “Constructive Approximation” and taught at the Universities of Kaiserslautern and Siegen. More and more students have encouraged me to turn my lecture notes into a textbook; hence, when Birkhäuser asked me if I had plans to write a book, I decided to accept the students’ advice. The more I thought about the project, the more ideas I had about what else could be added and what could be presented in a different way than in my lectures. Although I ran the risk of turning the book project into a never-ending story, I finally managed to finish it. So, here it is.

As a textbook, this book cannot cover the whole area of constructive approximation on the real line, the sphere, and the ball. The one-dimensional part is kept brief because there are many books that already present this area in detail. The main purpose is to demonstrate the features of Fourier, spline, and wavelet methods so that analogues in the multivariate case become clear. The parts on the sphere and the ball concentrate on tools that have already been successfully applied to geophysical problems or problems of medical imaging. Alternative approaches and additional theoretical and practical achievements are also mentioned, and corresponding references are given.

This book addresses several distinct readers:

- Undergraduate students in their last year as well as graduate students of mathematics will hopefully find this book a helpful companion while they are attending courses in constructive approximation, harmonic analysis, numerical analysis, or spline and wavelet methods. I have purposely not written as a research monograph. Advanced experts will consider some explanations trivial, but students who are not familiar with spherical analysis, in particular, might profit from additional help given in the derivations.
- Students in geoscientific studies with a focus on mathematical methodologies are provided with an introduction to the fundamental numerical methods for approximating functions on the sphere and the ball. This book treats classical

global approximations by orthogonal polynomials (spherical harmonics) as well as modern localized methods based on splines, wavelets, and Slepian functions with a view to sparse regularization.
- Geoscientists who realize that they need to learn more about advanced approximation methods for their problems.
- Mathematicians facing an application where they, for example, need to approximate an unknown function on a sphere or a ball can use this textbook to learn how the presented approximation methods work and what the current state of the art of these tools is.
- Geomathematicians who are already familiar with constructive approximation for geoscientific applications will, hopefully, find some new insights into well-known concepts. They will also find references to further advanced results and additional publications.

This book would not exist without the help of several people, listed in no particular order. The first courses that I taught on constructive approximation basically summarized Willi Freeden's achievements on spherical approximation methods at that point in time. So, without him, there wouldn't have been such a course and, consequently, there wouldn't be the book that you are holding in your hands. For the last ten years, I have added further topics. Besides the fact that I am starting now with a brief introduction to one-dimensional approximation, I have added some of the results of constructive approximation on the three-dimensional ball that I obtained in cooperation with my own research group. For this reason, I want to thank Nahid Akhtar, Muhammad Akram, Abel Amirbekyan, Paula Berkel, Doreen Fischer, and Dominik Michel for their courage to tackle complicated inverse problems in three dimensions and their valuable contributions to my own long-range research project. They all left their footprints in this book. Successful research in geomathematics and many other scientific areas is nowadays only possible with a highly qualified and ambitious group of scientists.

I also want to thank those who proofread this book and (in addition to catching typing errors) gave numerous suggestions for improvements. They are Nicole Dröge, Doreen Fischer, Willi Freeden, and Roger Telschow. I am also thankful to the anonymous reviewers for giving useful comments and catching some more minor errors. Furthermore, I would like to express my gratitude to Frederik J. Simons for his comments regarding the section on Slepian functions. Moreover, I am grateful to so many students who have attended my lectures for more than ten years. Their feedback was always valuable, and without their encouragement I would have never written this textbook. In this context, my gratitude also goes to a series of PhD students who organized tutorials that accompanied my lectures and assisted the students in learning the subject. Furthermore, I also want to thank Tom Grasso and Ben Cronin from Birkhäuser for their advice and their patience with an author who didn't meet the deadline. Above all, special thanks go to my wife, Bärbel Michel,

for assisting me by typing the whole book, deciphering my handwriting, catching errors during the typing, and coping with stylistic requirements.

I hope that this book will be helpful to many students and scientists, and I appreciate any feedback.

Siegen, Germany Volker Michel

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