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Gitta Kutyniok • Demetrio Labate
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Shearlets

Multiscale Analysis for Multivariate Data

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time–frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

<i>Antenna theory</i>	<i>Prediction theory</i>
<i>Biomedical signal processing</i>	<i>Radar applications</i>
<i>Digital signal processing</i>	<i>Sampling theory</i>
<i>Fast algorithms</i>	<i>Spectral estimation</i>
<i>Gabor theory and applications</i>	<i>Speech processing</i>
<i>Image processing</i>	<i>Time–frequency and</i>
<i>Numerical partial differential equations</i>	<i>time-scale analysis</i>
	<i>Wavelet theory</i>

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries, Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of “function”. Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantors set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, e.g., by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener’s Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in timefrequency-scale methods such as wavelet theory. The

coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the *raison d'être* of the *ANHA* series!

University of Maryland
College Park

John J. Benedetto
Series Editor

Preface

The introduction of wavelets about 20 years ago has revolutionized applied mathematics, computer science, and engineering by providing a highly effective methodology for analyzing and processing univariate functions/signals containing singularities. However, wavelets do not perform equally well in the multivariate case due to the fact that they are capable of efficiently encoding only isotropic features. This limitation can be seen by observing that Besov spaces can be precisely characterized by decay properties of sequences of wavelet coefficients, but they are not capable of capturing those geometric features which could be associated with edges and other distributed singularities. Indeed, such geometric features are essential in the multivariate setting, since multivariate problems are typically governed by anisotropic phenomena such as singularities concentrated on lower dimensional embedded manifolds. To deal with this challenge, several approaches were proposed in the attempt to extend the benefits of the wavelet framework to higher dimensions, with the aim of introducing representation systems which could provide both optimally sparse approximations of anisotropic features and a unified treatment of the continuum and digital world. Among the various methodologies proposed, such as curvelets and contourlets, the shearlet system, which was introduced in 2005, stands out as the first and so far the only approach capable of satisfying this combination of requirements.

Today, various directions of research have been established in the theory of shearlets. These include, in particular, the theory of continuous shearlets—associated with a parameter set of continuous range—and its application to the analysis of distributions. Another direction is the theory of discrete shearlets—associated with a discrete parameter set—and their sparse approximation properties. Thanks to the fact that shearlets provide a unified treatment of the continuum and digital realm through the utilization of the shearing operator, digitalization and hence numerical realizations can be performed in a faithful manner, and this leads to very efficient algorithms. Building on these results, several shearlet-based algorithms were developed to address a range of problems in image and data processing.

This book is the first monograph devoted to shearlets. It is not only aimed at and accessible to a broad readership including graduate students and researchers in the

areas of applied mathematics, computer science, and engineering, but it will also appeal to researchers working in any other field requiring highly efficient methodologies for the processing of multivariate data. Because of this fact, this volume can be used both as a state-of-the-art monograph on shearlets and advanced multiscale methods and as a textbook for graduate students.

This volume is organized into several tutorial-like chapters which cover the main aspects of theory and applications of shearlets and are written by the leading international experts in these areas. The first chapter provides a self-contained and comprehensive overview of the main results on shearlets and sets the basic notation and definitions which are used in the remainder of the book. The topics covered in the remaining chapters essentially follow the idea of going from the continuous setting, i.e., continuous shearlets and their microlocal properties, up to the discrete and digital setting, i.e., discrete shearlets, their digital realizations, and their applications. Each chapter is self-contained, which enables the reader to choose his/her own path through the book. Here is a brief outline of the content of each chapter.

The first chapter, written by the editors, provides an introduction and presents a self-contained overview of the main results on the theory and applications of shearlets. Starting with some background on frame theory and wavelets, it covers the definitions of continuous and discrete shearlets and the main results from the theory of shearlets, which are subsequently discussed in detail and expanded in the following chapters.

In the second chapter, Grohs focusses on the continuous shearlet transform. After making the reader familiar with concepts from microlocal analysis, he shows that the shearlet transform offers a simple and convenient way to characterize wavefront sets of distributions.

In the third chapter, Guo and Labate illustrate the ability of the continuous shearlet transform to characterize the set of singularities of multivariate functions and distributions. These properties set the groundwork for some of the imaging applications discussed in the eighth chapter.

In the fourth chapter, Dahlke et al. introduce the continuous shearlet transform for arbitrary space dimension. They further present the construction of smoothness spaces associated to shearlet representations and the analysis of their structural properties.

In the fifth chapter, Kutyniok et al. provide a comprehensive survey of the theory of sparse approximations of cartoon-like images using shearlets. Both the band-limited and the compactly supported shearlet frames are examined in this chapter.

In the sixth chapter, Sauer starts from the classical concepts of filterbanks and subband coding to present an entirely digital approach to shearlet multiresolution. This approach is not a discretization of the continuous transform, but is naturally connected to the filtering of digital data.

In the seventh chapter, Kutyniok et al. discuss the construction of digital realizations of the shearlet transform with a particular focus on a unified treatment of the continuum and digital realm. In particular, this chapter illustrates two distinct numerical implementations of the shearlet transform, one based on band-limited shearlets and the other based on compactly supported shearlets.

In the eighth chapter, Easley and Labate present the application of shearlets to several problems from imaging and data analysis to date. This includes the illustration of shearlet-based algorithms for image denoising, image enhancement, edge detection, image separation, deconvolution, and regularized reconstruction of Radon data. In all these applications, the ability of shearlet representations to handle anisotropic features efficiently is exploited in order to derive highly competitive numerical algorithms.

Finally, it is important to emphasize that the work presented in this volume would not have been possible without the interaction and discussions with many people during these years. We wish to thank the many students and researchers who over the years have given us insightful comments and suggestions, and helped this area of research to grow into its present form.

Berlin, Germany
Houston, USA

Gitta Kutyniok
Demetrio Labate

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