Applied and Numerical Harmonic Analysis

Ole Christensen

Functions, Spaces, and Expansions

Mathematical Tools in Physics and Engineering

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods.

The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

John J. Benedetto
Series Editor
University of Maryland
College Park

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Preface

The purpose of this book is to present some mathematical tools that play key roles in mathematics as well as in applied mathematics, physics, and engineering. The treatment is mathematical in nature, and we do not go into concrete applications; but it is important to stress that all the considered topics are selected because they actually play a role outside pure mathematics. The hope is that the book will be useful for students in many fields of science and engineering, and professionals who want a deeper insight in some of the topics appearing in the scientific literature.

A central theme throughout the work is the structure of various vector spaces (most importantly, normed vector spaces and Hilbert spaces) and expansions of elements in these spaces in terms of bases. Particular attention is given to the space of square-integrable functions, $L^2(\mathbb{R})$.

The goal is twofold. Besides the interest in these subjects by themselves, the book will also contribute to a deeper understanding of several themes from calculus and linear algebra, because these themes appear here again and are tied together. For example, we discuss Fourier series in the correct setting of an expansion in a Hilbert space, similar to the one that is obtained via an orthonormal basis in $\mathbb{C}^n$.

Before we go into detail about the content of the book, let us spend a few lines on the prerequisites. We expect the reader to

- Have a profound understanding of linear algebra, as well in $\mathbb{R}^n$ and $\mathbb{C}^n$ as in general vector spaces;
- Be familiar with the basic concepts of calculus and real analysis, including (Riemann) integration and infinite series of real or complex numbers.

The core of the book is formed by Chapters 2–7. Chapter 1 is a survey on topics from elementary mathematics courses, and Chapters 8–11 describe concrete functions and settings where the key concepts treated in Chapters 2–7 play a central role. Each chapter ends with a collection of exercises.

Let us describe the content in more detail. Chapter 1 collects some basic results from linear algebra and calculus, e.g., concerning topology in $\mathbb{C}^n$ and continuity of functions. We expect the reader to be familiar with most of the topics in this chapter. All results are stated without proofs, but in many cases a guide to a proof can be found in the exercises.

Chapters 2 and 3 deal with particular types of vector spaces on an abstract level. The aim is a detailed mathematical description. All results are presented either with a proof or the proof left as an exercise. In the first of these chapters, Chapter 2, the key concept of a normed vector space is presented. We discuss linear operators on such spaces, and infinite series consisting of vectors in normed spaces. Chapter 3 deals with Banach spaces, in particular the sequence spaces $\ell^p(\mathbb{N})$, and operators hereon. Chapter 4 specializes in the important case of a norm arising from an inner product. This leads to the concept of a Hilbert space. We continue the analysis of linear operators initiated in Chapter 3, now with focus on results that are particular for Hilbert spaces. The key concept of an orthonormal basis is introduced.

While Chapters 2–4 are abstract in nature, Chapter 5 marks the beginning of a more concrete part of the book. There is still emphasis on the mathematical formalism. However, we do not insist on a complete treatment: we skip discussions of certain technical issues, and some results are presented without a proof.

In Chapter 5 we consider an important class of Banach spaces consisting of functions, the so-called L^p-spaces. Special emphasis is given to the space $L^1(\mathbb{R})$ and integration techniques on that space. Chapter 6 specializes in the case $p = 2$, which leads to a Hilbert space. We consider various operators on $L^2(\mathbb{R})$. We also consider L^2-spaces on an interval, and relate these spaces to Fourier analysis. Chapter 7 deals with the Fourier transform, convolution, and the sampling problem.

The final part of the book discusses special classes of functions that appear in many areas of applied mathematics and are related to the themes presented in the book. All the considered functions naturally lead to bases for $L^2(\mathbb{R})$ or subspaces hereof. Chapter 8 provides a short description of wavelet theory in $L^2(\mathbb{R})$, based on Fourier analysis. A more detailed analysis of the key tool in wavelet theory, multiresolution analysis, is given in Chapter 9. Chapter 10 introduces the important B-splines and their main properties. In Chapter 11 we consider certain functions (typically polynomials) that arise as solutions to various differential equations appearing, for example, in physics. It turns out that for special differential equations, the collection of some particular solutions form an orthonormal basis for a certain L^2-space; that brings us back to the main theme of the book.

Appendix A collects certain proofs that are particularly long and technical. Appendix B contains a list of the vector spaces considered in the book and their main properties, as well as a list of some of the special functions considered in Chapter 11.

For use in a course at the master's level, the natural starting point is Chapter 2; to the extent that the results in Chapter 1 are unknown, they can be presented section by section during the course whenever relevant. Depending on the anticipated content of the course, one can proceed in various ways. For a course focusing on Hilbert spaces, one can skip most of Chapter 3 (except the definition of a Banach space) and move directly to Chapters 4–7; on the other hand, a profound understanding of general Banach spaces requires inclusion of Chapter 3. Concrete manifestations of the abstract concept of an orthonormal basis appear in Chapters 8–10 (wavelets, in particular, based on B-splines) and Chapter 11 (orthonormal bases consisting of solutions to special differential equations).

The list of references contains articles and books at several levels. In order to be more informative, we have introduced the following ranking system to the references: **(A)** elementary; **(B)** undergraduate level; **(C)** graduate level; **(D)** research paper; **(H)** historical paper.

I would like to thank Robert Burckel and Christopher Heil for many constructive comments to an earlier version of the manuscript. Their help greatly improved the presentation. I also thank the many students at the Technical University of Denmark who helped me by finding print errors and spotting unclear formulations in the preliminary manuscripts I used in the spring semesters of 2008 and 2009. Finally, I thank the staff at Birkhäuser, especially Tom Grasso and Patrick Keene, for their help and careful copyediting of the manuscript.

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Prologue: Spaces and Expansions

In brief, the content of this book is captured by the two themes *spaces* and *expansions.* The purpose of this prologue is to relate the key topics to physics and engineering.

In engineering and signal processing, a *signal* means a *function* f, typically with the time as variable. For example, the signal might be the current running in the loudspeaker cable when a certain recording is played. Such a signal is shown in Figure 1. In order to extract relevant features in the signal, the signal is often considered in a *transformed domain:* in the example with the recording, if we want to extract information about the frequencies appearing in the signal, one would consider the *Fourier transform* of the signal. The Fourier transform of f is formally defined as the function

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} \, dx, \ \gamma \in \mathbb{R}. \tag{1}$$

For the function in Figure 1, the absolute value of the Fourier transform is depicted in Figure 2. It shows that the signal has a large content of frequencies around 500 Hz, which is close to the frequency 440 Hz that is used to tune the instruments in an orchestra.

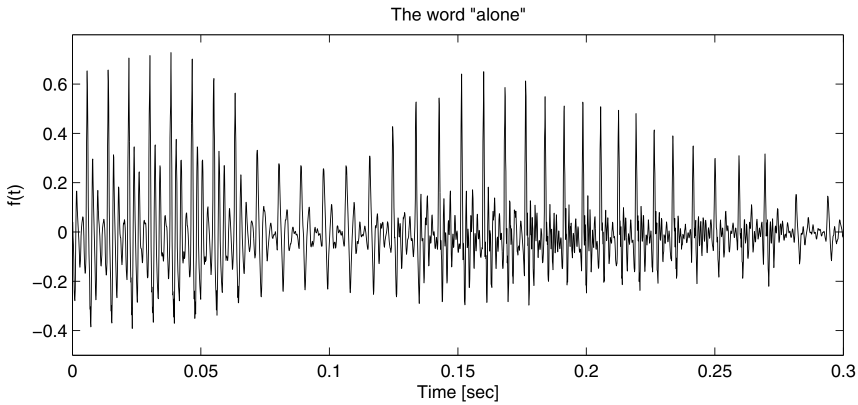

Figure 1. A speech signal. One can regard such a signal as the current in the cable to the loudspeaker when a recording of the speech is played. The actual signal is a recording of the word “alone”.

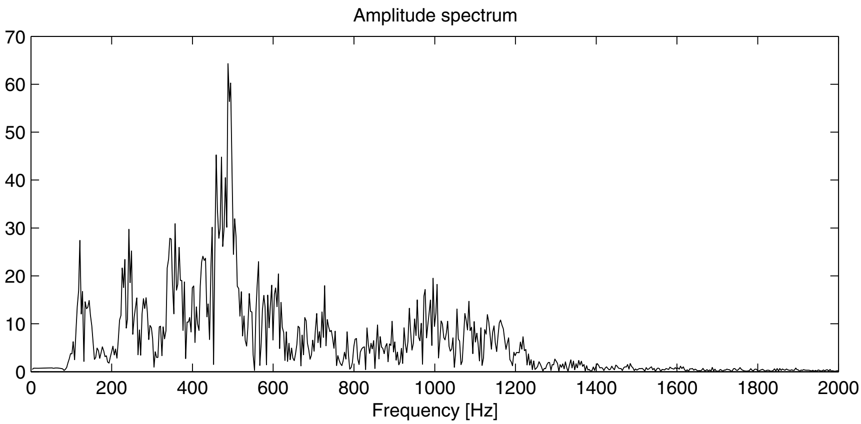

Figure 2. The absolute value of the Fourier transform of the signal in Figure 1.

However, the expression (1) only makes sense under certain restrictions on the function f. In other words: we have to specify the applicable signals. Mathematically, this is done by requiring all considered signals f to belong to certain *vector spaces:* in the concrete case discussed here, the relevant spaces are the (Banach) spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. These spaces play a central role in the book.

The concept *an expansion* is known from elementary linear algebra: it is merely another name for a representation of a vector $\mathbf{v}$ in a vector space in terms of a basis $\{\mathbf{e}_k\}$,

$$\mathbf{v} = \sum c_k \mathbf{e}_k.$$

We know that the choice of a convenient basis is crucial: for example, the representation of a linear operator might be very complicated with respect to some unfortunate bases, but very easy with respect to a well-chosen basis. One of the key concepts of the book is to discuss bases and expansions in infinite-dimensional vector spaces. Readers having knowledge about quantum mechanics and coherent states already know about series expansions in terms of eigenfunctions for certain differential equations; such expansions appear as special cases of the general theory presented in this book.

Already in Example 1.1.3 we will introduce the discrete Fourier transform basis for $\mathbb{C}^n$. Paying close attention to that example will help the reader to see the motivation behind much of the material to be presented later in the book.