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Topics in Operator Semigroups

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To Ita, Bracha, Pnina, Pinchas, and Ruth

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Preface

This book is based on lecture notes from a second-year graduate course, and is a greatly expanded version of our previous monograph [K8]. We expose some aspects of the theory of semigroups of linear operators, mostly (but not only) from the point of view of its meeting with that part of spectral theory which is concerned with the integral representation of families of operators. This approach and selection of topics differentiate this book from others in the general area, and reflect the author's own research directions. There is no attempt therefore to cover thoroughly the theory of semigroups of operators. This theory and its applications are extensively exposed in many books, from the classic Hille–Phillips monograph [HP] to the most recent textbook of Engel and Nagel [EN2] (see [A], [BB], [Cl], [D3], [EN1], [EN2], [Fat], [G], [HP], [P], [Vr], and others), as well as in chapters in more general texts on Functional Analysis and the theory of linear operators (cf. [D5], [DS I–III], [Kat1], [RS], [Y], and many others). Nevertheless, because the book is based on a course, and because we intended to make it reasonably self-contained and convenient both for independent study and for a graduate course or seminar, we have included in Section A of Part I (making it thereby the longest section of the book!) an exposition of the basic theory: the classical Hille–Yosida theory on the interplay between a semigroup and its generator up to the characterization of the generator of a (strongly continuous) semigroup by means of estimates on the resolvent iterates, the Lumer–Phillips theory of dissipative operators with its "resolvent-free" characterization of the generator, the Trotter–Kato convergence theorem on the equivalence of "graph convergence" of generators and "strong convergence" of the corresponding semigroups, the Kato unified treatment of the "exponential formula" and the "Trotter product formula," and the Hille–Phillips perturbation theorem for generators of C_o-semigroups. As a transition to the "integral representations" mentioned above, we conclude this section with Stone's theorem for (semi)groups of *unitary* operators and Sz.-Nagy's spectral integral representation for *bounded groups* of operators in Hilbert space.

In Section B of Part I, we construct the *semi-simplicity space* for a given C_o-group of operators in Banach space. It is a Banach subspace which is maximal for the existence of a spectral integral representation of the group on it.

In Section C, we are concerned with *analytic semigroups*, that is, semigroups that possess an analytic continuation to some sector in the complex plane. We present an approach independent of contour integrals, that yields easily to characterizations of the generators of such semigroups.

The semigroup is considered as a *function of its generator* in Section D. We prove a "noncommutative Taylor formula," and consider families of semigroups whose generators depend analytically on a complex parameter in a natural sense. The conceptual meaning of the latter analysis is the hereditary property of analyticity from the coefficients of an Abstract Cauchy Problem to its solution.

The *asymptotic behavior* of (one-parameter) semigroups for large values of the parameter is taken up in Section E. We first consider the relatively simple case of analytic semigroups and of various kinds of "averages" of a semigroup, which include as special cases its Cesaro, Abel, and Gauss averages. We then prove the Arendt–Batty–Lyubich–Vu ("ABLV") stability theorem, using the technique of the so-called "asymptotic space." Adequate conditions on the spectrum of the generator insure the (strong) "stability" of the semigroup, that is, the latter's strong convergence to zero when the parameter tends to infinity. Additional results on stability are included in the "Miscellaneous Exercises" section at the end of the book.

In Section F, we obtain a characterization of generators of *regular semigroups*, that is, analytic semigroups in the right halfplane that possess boundary values on the imaginary axis. We then proceed with the analysis of some classical examples.

A brief discussion of *pre-semigroups*, also called "C-semigroups" or "regularized semigroups" in the literature, concludes Part I of the book (Section G). Pre-semigroups were introduced in germinal form in [DaP], and their extensive study was started in [DP]. They play a role in the solution of the abstract Cauchy problem for an operator which is not necessarily the generator of a semigroup, and is not even densely defined. (The monograph [DL4] presents the theory in great detail, as well as many applications to partial differential equations.)

In Part II, we turn to a more detailed study of integral representations in the spirit of Section B of Part I.

In Section A, the semi-simplicity space is constructed for (generally unbounded) operators *that are not necessarily semigroup generators*, provided they have real spectrum, or at least have a half-line in their resolvent set. A spectral integral representation is obtained for the part of the given operator in its semi-simplicity space, and the latter is a maximal Banach subspace with this property.

In an analogous manner, the *Laplace–Stieltjes space* and the *integrated Laplace space* for a family of closed operators are constructed in Section B by an adequate *renorming method.* As applications, we obtain a spectral integral representation for semigroups of *closed* operators, and a characterization of generators of *n-times integrated semigroups.*

Section C takes up the spectral integral representation for families of unbounded symmetric operators in Hilbert space, defined only *locally* (with respect to the parameter) in a suitable sense. We present the Frohlich–Klein–Landau theory of *local semigroups of unbounded symmetric operators*, generalizing the classical Stone theorem, and an analogous theory for *cosine families of unbounded symmetric operators.* These theories provide a natural approach to Nelson's *Analytic Vectors Theorem* and to Nussbaum's *Semi-analytic Vectors Theorem*, respectively.

Part III contains a small dose of applications, selected from the vast material in the literature by the criterion of our own involvement in their derivations. As mentioned at the beginning of this Introduction, our choice avoids overlapping with the existing monographs dealing with applications of operator semigroup theory in areas such as Markov processes, the Abstract Cauchy Problem, evolution equations, Mathematical Physics, etc. We refer the interested reader to the latter texts, some of which are listed in the Bibliography section.

In Section A of Part III, the results on analytic families of semigroups (exposed in Section D of Part I) are applied to the Abstract Cauchy Problem in the "temporally inhomogeneous" case. Under either Kato's or Tanabe's conditions, it is shown that "coefficients analyticity" implies "solutions analyticity" (with respect to an auxiliary complex parameter).

In Section B, we apply the results of Sections A and F of Part I to the analysis of similarity within the family of operators $S + \zeta V$ (where ζ is a complex parameter), when iS generates a C_o-group $S(\cdot)$, and V is a bounded operator satisfying with S the so-called *Volterra commutation relation* $[S, V] \subset V^2$. This study is motivated by the classical pair of operators on $L^p(0,1)$, $1 < p < \infty$, defined by $S : f(x) \to xf(x)$ and $V : f(x) \to \int_0^x f(s)\,ds$. In this latter case, $S + \zeta V$ is similar to $S + \omega V$ if and only if $\Re\zeta = \Re\omega$ (cf. [K19]). In the abstract situation (under some additional condition on V), $S + \zeta V$ is similar to S if and only if $\Re\zeta = 0$. Thus, in particular, $S - V$ is *not* similar to S. However, it is proved in the last subsection that the perturbations $(S - V) + P$ *are* similar to S for all P in the "similarity suborbit" $\{S(-t)VS(t);\ t \in \mathbb{R}\}$ of V.

A collection of "exercises" is appended to the main text. In many cases, the exercise contains a significant result, which is reached through the given sequence of steps.