BIRKHÄUSER

Applied and Numerical Harmonic Analysis

Brigitte Forster • Peter Massopust
Editors

Four Short Courses on Harmonic Analysis

Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis

With Contributions by

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

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Preface

Classical harmonic analysis studies problems related to series expansions of signals or functions using trigonometric polynomials. The theory of Fourier series and Fourier integrals forms the core of harmonic analysis and extends from there to other mathematical areas such as the theory of singular integrals, approximation theory, and sampling theory, just to mention a few. Harmonic analysis is also used in numerous applications where it can be thought of as the mathematical backbone for a large number of modern methods in signal analyis and signal processing as well as image analysis and image processing. Its internal growth has seen generalizations to nontrigonometric expansions and noncommutative group settings, but its basic role in other areas of mathematics (differential equations, number theory, probability theory, and statistics), physics and chemistry (wave phenomena, crystallography, and optics), financial analysis (time series), medicine (tomography, brain and heart wave analyses), and biological signal processing has made harmonic analysis the main fundamental contributor to all of 20th century's human-based technologies. These include telephone, radio, television, radar and sonar, satellite and wireless communications, medical imaging, the Internet, and multimedia.

The applications of harmonic analysis to medical image processing have been undergoing a rapid change primarily driven by better hardware and software. Part of this development is an attempt by researchers to base medical engineering principles on solid and rigorous mathematical foundations, and to develop mathematical methods that allow the creation of effective software programs that reduce or replace invasive medical procedures.

Approximation theory and harmonic analysis benefit from each other. The latter provides the means that the former uses to approximate complicated functions or signals and surfaces or images, and to estimate the errors of this approximation. On the other hand, harmonic analysis problems often require methods or input from approximation theory. Like harmonic analysis, approximation theory has seen decades of rapid development and growth, again, primarily driven by applications, such as computer-aided geometric design (CAGD) and its various ramifications.

Recently, a great deal of emphasis has been put into the digitization, transmission, and processing of three-dimensional data sets. One-dimensional methods developed in harmonic analysis and approximation theory in the past do not easily carry over to this higher-dimensional setting. Instead, new ideas and methods need to be found to take into account the nonisotropy and nonhomogeneities inherent in such data sets. In order for these generalizations to take place, new ideas from lower-dimensional problems need to be reconsidered. As an example, we take the effective design of wave forms that is essential to the simultaneous transmission of clear messages on the same frequency band. Constructive approximations of unimodular sequences whose autocorrelations vanish on prescribed sets are introduced, and their analysis depends signifantly on Wiener's generalized harmonic analysis (see [19]).

Signal analysis and image analysis have greatly benefited from the theory of wavelets and their generalizations to frames. These multiscale methods use representations based on two specific groups that are used to transfer information between the scales and within each scale. It has become clear that for multidimensional data, more general groups and multiscale methods need to be employed. The geometry involved in such a high-dimensional setting is more complicated and challenging than in the one-dimensional case, as spatial and, in the video setting, even temporal features need to be taken into account. A first step toward such an improvement in representation is undertaken in [130, 233].

This advanced textbook is intended for graduate students, pure and applied mathematicians, mathematical physicists, and engineers working in image/signal processing and communication theory. The book may be used in an advanced topics course or in a seminar on harmonics analysis and its applications to image and signal analysis. The prerequisites are a solid background in linear algebra and real analysis and knowledge of the fundamentals of functional analysis and metric topology.

Chapters 2, 3, 4, and 5 in this book are based on lectures given by their authors at the summer school on New Trends and Directions in Harmonic Analysis, Approximation Theory, and Image Analysis, which took place in Inzell, Germany, from September 17–21, 2007. One of the goals of this summer school was to bring together a distinguished group of highly established international researchers to present their latest cutting-edge research, and, in conjunction with a small group of scientists including young researchers, to establish new and exciting directions for future investigation into the topics described above.

A short introduction to the mathematical aspects of time-frequency analysis paves the way for the above-mentioned chapters. The reader is exposed to the main themes presented in this book and provided with a summary of those mathematical notions and concepts needed to fully appreciate the contents of Chapters 2 to 5. In addition, the material in these chapters is put into perspective in this introductory chapter.

Chapters 2 to 5 were written by internationally renowned mathematicians and have an expository and interdisciplinary character, allowing the reader to understand the theory behind modern image and signal processing methodologies. In detail, the chapters cover the following.

Ole Christensen considers B-spline generated frames. He exploits the flexibility of frames and combines them with the elegant representations for B-splines. In the first part of his chapter, he introduces the terminology of Bessel sequences, Riesz bases, and frames and exhibits their central properties. In the second part, he considers concrete constructions for Gabor systems and other tight frames, before he finally deduces the wavelet frames generated by B-splines via the so-called unitary extension principle.

Demetrio Labate and Guido Weiss consider the theory and applications of composite wavelets. They first describe the unified theory of reproducing systems, a simple and flexible mathematical framework to characterize and analyze wavelets, Gabor systems, and other reproducing systems in a unified manner. These systems can be rewritten as a countable family of translations applied to a countable collection of functions. The authors then define wavelets with composite dilations, a novel class of reproducing systems that provide truly multidimensional generalizations of traditional wavelets, and discuss so-called shearlets as a special case of optimally sparse representations for 2D. Applications in edge detection and considerations on the continuous analogues of composite wavelets are also considered.

Pierre Vandergheynst and Yves Wiaux introduce wavelets on the sphere and therefore leave the classical Cartesian space. For many applications such as astrophysics, geophysics, neuroscience, computer vision, and computer graphics, data are given as functions on the sphere. In all these situations, one is compelled to design data analysis tools that are adapted to spherical geometry, for one cannot simply project the data into Euclidean geometry without having to deal with severe distortions. The authors provide a generalization of the wavelet transform to signals on the sphere. This generalization is not trivial, as the dilation operator is not well defined on the sphere. In addition, any algorithm faces the problem of how to sample data on the sphere. This chapter discusses some recently developed methods for the analysis and reconstruction of signals on the sphere with wavelets, on the basis of theory, implementation, and applications.

Karlheinz Gröchenig gives various new and interesting aspects of Wiener's Lemma. This result is one of the main theorems of Banach algebra theory. In the first part of his chapter, he discusses Wiener's Lemma in detail and investigates equivalent formulations for convolution operators. In the second part, he considers various variations, especially in noncommutative settings. He also shows the importance of the lemma for time-varying systems and pseudodifferential operators and concludes with applications in mobile communications.

One of the main features of this book is its emphasis on the interdependence of these four modern research directions. Each chapter ends with exercises that allow for a more in-depth understanding of the material and are intended to stimulate the reader to further research.

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