BIRKHÄUSER

**Progress in Mathematics**
Volume 276

Claudio Bartocci
Ugo Bruzzo
Daniel Hernández Ruipérez

# Fourier–Mukai and Nahm Transforms in Geometry and Mathematical Physics

Birkhäuser
Boston • Basel • Berlin

Claudio Bartocci
Dipartimento di Matematica
Università di Genova
Genova, Italy
bartocci@dima.unige.it

Ugo Bruzzo
Scuola Internazionale Superiore di
Studi Avanzati and Istituto Nazionale
di Fisica Nucleare
Trieste, Italy
bruzzo@sissa.it

Daniel Hernández Ruipérez
Departamento de Matemáticas
and Instituto Universitario de Fisica
Fundamental y Matemáticas
Universidad de Salamanca
Salamanca, Spain
ruiperez@usal.es

ISBN 978-0-8176-3246-5 e-ISBN 978-0-8176–4663-9
DOI 10.1007/b11801

Library of Congress Control Number: 2009926479

Mathematics Subject Classification (2000): 14-02, 14D21, 14D20, 14E05, 14F05, 14J28, 14J32, 14J81, 14K05, 18E30, 19K56, 53C07, 58J20



Cover Design by Joseph Sherman

Printed on acid-free paper

Birkhäuser Boston is part of Springer Science+Business Media (www.springer.com)

# Contents

# Preface

A fundamental question in geometry is to find invariants for a given class of geometric objects. In the case of algebraic varieties, natural invariants are the Chow groups and the algebraic K-theory. In looking for finer invariants one could think of the category of coherent sheaves; however, it is known that two projective varieties are isomorphic if and only if the respective categories of coherent sheaves are equivalent [120]. A straightforward extension of this idea is to look at the derived category of coherent sheaves. That notion was used by Grothendieck and Verdier as an appropriate framework for their theory of duality; moreover, in the late 1970s Beĭlinson gave a simple characterization of the derived categories of projective spaces. Derived categories of coherent sheaves appeared again in the fundamental paper by Shigeru Mukai [224], where the integral functor now called "Fourier-Mukai transform" was introduced.

A radical change of perspective in connection with derived categories took place with a result of Bondal and Orlov, according to which the derived category of a projective variety, whose canonical or anticanonical bundle is ample, fully determines the variety. Subsequent work by Orlov showed that any equivalence of derived categories of coherent sheaves on two projective varieties $X$ and $Y$ is an integral functor — i.e., a kind of "correspondence" induced by an object in the derived category of the product $X \times Y$. This prompts us to consider the more general question: to what extent does the derived category of coherent sheaves determine the underlying algebraic variety? And also, what is the relationship between the group of automorphisms of the derived category of a projective variety and the group of isomorphisms of the variety?

In the second half of the 1990s, physicists working in string theory got interested in triangulated categories. Actually, the quantum theories corresponding to some string models admit solitonic states (branes), which can be geometrically characterized as coherent sheaves supported on subvarieties of the compactification space of the string (in most cases a Calabi-Yau threefold). Thus, derived categories of coherent sheaves naturally come into play, and integral functors can be exploited to describe some mirror dualities.

This book is about the study of integral functors. Given two projective varieties $X$ and $Y$, an integral functor between the derived categories $D(X)$ and $D(Y)$ is a functor of the type

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)\,,$$

where $\mathcal{K}^\bullet$ is an object in $D(X \times Y)$ (a complex of coherent sheaves on $X \times Y$, called the *kernel* of the integral functor) and $\pi_X$, $\pi_Y$ are the projections onto the two factors of $X \times Y$ (for precise definitions, the reader is referred to Chapter 1). When an integral functor is an equivalence of categories, we shall call it a *Fourier-Mukai functor*, and if in addition the kernel $\mathcal{K}^\bullet$ is concentrated (i.e., it reduces to a single coherent sheaf on $X \times Y$), the integral functor will be called a *Fourier-Mukai transform.* The prototype of this kind of transform was defined by Mukai in 1981: $X$ is an Abelian variety, $Y$ its dual variety, and the kernel is the Poincaré bundle on the product $X \times Y$.

Besides developing the basic theory of integral functors, we shall put special emphasis on some of their applications to geometry and mathematical physics. As a matter of fact, one proves that whenever two projective varieties $X$ and $Y$ have equivalent derived categories (i.e., they are *Fourier-Mukai partners*), then $Y$ is a coarse moduli space of coherent sheaves on $X$, and vice versa. Thus, we shall be mostly concerned with applications to moduli spaces of sheaves. A first example is Mukai's original transform, which establishes the equivalence between the derived categories of an Abelian variety $X$ and of its dual variety $\hat{X}$, which is indeed the moduli space of flat line bundles on $X$. Other examples are provided by the construction of Fourier-Mukai transforms for K3 surfaces, and by the relative Fourier-Mukai transforms. The latter play an important role in the so-called homological mirror symmetry in string theory and in some constructions appearing in the theory of algebraically completely integrable systems.

We now give a cursory presentation of the contents of the book. Chapter 1 is devoted to the foundations of the theory of integral functors. A key notion is that of strongly simple object, which provides a characterization of fully faithful integral functors (Theorem 1.27). This result — due to Bondal and Orlov [48] — will be a cornerstone on which we shall build many of the most fundamental theorems proved in this book. In Chapter 2 we further develop the theory of these functors, considering the case when they are equivalences of derived categories. Orlov's representability theorem [242] — whose proof involves the notion of spanning classes for a derived category of coherent sheaves — shows that every equivalence of derived categories is an integral functor. This explains the pervasiveness of Fourier-Mukai functors in the study of the geometry of algebraic varieties.

While the two initial chapters are framed in a rather general setting, in Chapters 3 and 4 we study in some detail the cases of Abelian varieties and K3 surfaces. In particular, in Chapter 3 we review Mukai's original construction and present some applications. The first nontrivial instance of Fourier-Mukai transform was obtained on K3 surfaces by the current authors [24]. In this case, $X$ is a K3 surface whose Néron-Severi group satisfies certain restrictions, $Y$ is 2-dimensional compact component of the moduli space of stable bundles over $X$ (which one proves to be a K3 surface as well) and the kernel is the relevant universal sheaf. A remarkable feature of the Fourier-Mukai transform in the case of both Abelian and K3 surfaces is that, under suitable assumptions, it preserves the stability of the sheaves it operates on. Consequently, it supplies a helpful tool to investigate the structure of moduli spaces of stables sheaves, as we show, e.g., in Section 4.5.

Chapter 5 is a digression in the realm of complex differential geometry. On a compact Kähler manifold, Hermitian-Yang-Mills bundles and stable bundles are related by the celebrated Hitchin-Kobayashi correspondence. Regarding an Abelian surface as a flat 4-dimensional real torus $T$ and applying this correspondence, Mukai's original transform translates into Nahm's transform, introduced in the early 1980s by the physicist Werner Nahm to study periodic instantons on $\mathbb{R}^4$ [230]. This transform is based on an index-theoretic construction: thinking of the dual torus $\widehat{T}$ as a space parameterizing a family of twisted Dirac operators, one can associate an index bundle to any instanton on $T$. After extending Nahm's transform to a more general setting, we show how it relates with a Fourier-Mukai transform. We examine in some detail the case when the base manifold carries a hyperkähler structure. Besides being interesting on its own, this perspective sheds new light on some results obtained in the algebro-geometric framework: for instance, in Section 5.4, we provide a different proof of the preservation of stability for bundles on Abelian and K3 surfaces.

In Chapter 6 we develop the machinery of relative Fourier-Mukai functors for algebraic $B$-schemes. These are a particular kind of integral functors, whose kernel is an element of the derived category of the fibered product $X \times_B Y$ (here $X$ and $Y$ are schemes over a base scheme $B$). We offer a quite comprehensive treatment of elliptic fibrations $X \to B$, dealing separately with the case when the fibration admits a Weierstraß model (allowing the base scheme to be of arbitrary dimension) and the case of relatively minimal elliptic surfaces. In the first situation, we study the moduli spaces of relatively semistable sheaves and discuss the notion of spectral cover. If the total space $X$ of the Weierstraß fibration has dimension 2 or 3, the Fourier-Mukai transform establishes a correspondence between relatively semistables sheaves on $X$ and spectral data on the compactified relative Jacobian associated with the fibration. When $X$ is an elliptically fibered Calabi-Yau threefold, this construction — originally obtained by Friedman, Morgan and Witten [113, 114] — is relevant to string theory.

The study of the Fourier-Mukai functor on elliptic surfaces is part of a much wider research program, which we pursue in Chapter 7. Two projective varieties are said to be Fourier-Mukai partners if there is an exact equivalence of triangulated categories between their bounded derived categories; in view of Orlov's representability theorem, this amounts to the existence of a Fourier-Mukai functor between them. Section 7.4 is devoted to the classification of Fourier-Mukai partners of algebraic surfaces, a result that for minimal surfaces was first obtained by Bridgeland and Maciocia. The problem of the birational invariance of the derived category of a projective variety is dealt with in Section 7.5, while Bridgeland-King-Reid's interpretation of the McKay correspondence is presented in Section 7.6. The results presented in Chapter 7 are a clear indication of the significance of Fourier-Mukai transforms in algebraic geometry, as it has been recently pointed out by Kawamata and others.

We have made an effort to be self-contained, devoting three appendices to present some preliminary material (respectively, derived categories, integral lattices and a miscellany of results in algebraic geometry), and proving most of the fundamental theorems from scratch. Generically, prerequisites for reading this book reduce to a basic knowledge of algebraic geometry at the level of Hartshorne [141] and, for Chapter 5, to some rudiments of differential geometry (manifolds, bundles, connections, and on a somehow more advanced level, some theory of elliptic operators on spaces of sections of a vector bundle). In addition, we use quite extensively some basic categorial language, and occasionally, we employ spectral sequences.

On the other hand, we did not aim at giving an exhaustive treatment of the subject, which appears to be widely ramified and steadily growing. Among the most conspicuous omissions, we do not deal with the case of Fourier-Mukai functors on singular varieties and we leave out the important topic of autoequivalences of derived categories. Furthermore, we mention only cursorily the recent developments related to differential graded categories and derived algebraic geometry. We have tried to put a remedy to major and minor omissions adding a "notes and further reading" section at the end of each chapter. In this connection one should also cite D. Huybrecht's book on Fourier-Mukai transforms [153]. Should one try to compare the two books in terms of their contents, one would notice that our book is more abundant in technical details, and somehow aims at a reasonably self-contained treatment of the arguments it touches. On the other hand, the choice of topics in the two books is somehow different; in this sense, we believe that the two books nicely complement each other.

A final appendix serves as an introduction to one of the most interesting recent developments related to integral functors, namely, the notion of stability condition for derived categories.

Claudio Bartocci
Ugo Bruzzo
Daniel Hernández Ruipérez

April 2008

# Acknowledgments

The project of writing this book saw the light in 2002 at a conference in Cookeville, Tennessee, organized by our friend Rafał Abłamowicz. It is very apt that he is the first person whom we thank. The project started with a fourth author, Marcos Jardim, who at some stage preferred to withdraw. We owe many thanks to Marcos: without him this book would not have been written.

Appendix A, an introduction to derived categories that originates from a set of notes for a course at the School on Algebraic Geometry and Physics in Salamanca in September 2003, has been written by Fernando Sancho. Appendix D, an introduction to stability conditions for derived categories, has been written by Emanuele Macrì. We are deeply thankful to Emanuele and Fernando for their contributions to this work.

Very special thanks are due to David Ploog, who was patient enough to read this book twice during the final stage of its redaction, pointed out several mistakes and inconsistencies, and proposed several improvements. Without David's help this book would definitely have been worse. We also thank Benoit Charbonneau, Stefano Guerra, Daniel Hernández Serrano, Adrian Langer, Cristina López Martín, Tony Pantev, Darío Sánchez Gómez, Justin Sawon, Edoardo Sernesi, Carlos Tejero Prieto and the anonymous referee for pointing out mistakes and suggesting improvements.

We have included many results obtained jointly with our collaborators Björn Andreas, Antony Maciocia, José M. Muñoz Porras and Fabio Pioli: we are grateful to all of them. We also thank Björn for clarifying some physics-related issues.

Many thanks are due to Birkhäuser, especially in the person of Ann Kostant, for the enthusiasm with which this project was considered, and for patiently waiting until it was over.

Writing this book has required many exchanges of visits among the authors (very likely, the most pleasant aspect of the enterprise). These have been made possible by funding provided by the University of Genova, the International School for Advanced Studies in Trieste and the University of Salamanca, by the Italian

National Research Projects (P.R.I.N.) “Geometric methods in the theory of nonlinear waves” and “Geometry on algebraic varieties,” and by the Spanish Research Projects “Geometrical integral transforms and applications” (BFM2003-00097), “Applications of integral functors to geometry and physics” (GCyL-SA114/04) and “Coherent sheaves, derived categories and integral functors in birational geometry and string theory” (MTM2006-04779). A strong initial impulse to the writing of the book was given by a “Research in Pairs” stay of the first and second author at Mathematische Förschunginstitut Oberwolfach, which we deeply thank, especially in the person of its director, Professor Martin Greuel. The editing of the book was finalized when the second author was spending the 2008 spring term at the Department of Mathematics of the University of Pennsylvania in Philadelphia. Warm thanks are due to the scientists and staff of the department for their collaboration and the nice atmosphere they create and to the university for providing support.