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Yoshio Sone

# Molecular Gas Dynamics

*Theory, Techniques, and Applications*



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# Preface

*Molecular Gas Dynamics* originates from lectures and seminars delivered by the author at various universities and institutions worldwide. These materials are supplemented and arranged in a form appropriate to a graduate textbook on molecular gas dynamics, or gas dynamics on the basis of kinetic theory. The book provides an up-to-date description of the basic theory of molecular gas dynamics and its various applications giving interesting and important gas dynamic phenomena. The progress of molecular gas dynamics in the last forty years has greatly enhanced the contents of the basic theory and provided information on various interesting and important gas dynamic problems. This has made it possible to compile a new graduate textbook on molecular gas dynamics. The present book reflects these developments providing working knowledge: theory, techniques, and typical phenomena in a rarefied gas (low-density and micro flows), for future theoretical development and applications.

The book begins with a brief presentation of the fundamental properties of the Boltzmann equation and a summary of notation used globally in subsequent chapters of the book. A full explanation of the fundamental properties is given in Appendix A. The author hopes that readers of various backgrounds can proceed quickly to the main subject, with reference to Appendix A if necessary. As is apparent from the table of contents, after presenting general theories for highly and slightly rarefied gases and various simple flows, such as unidirectional or quasi-unidirectional flows, and flows around a sphere, the author discusses various subjects: flows induced by temperature fields, which are typical in a rarefied gas; flows with evaporation and condensation; bifurcation of flows in a rarefied gas; and ghost effects in a gas in the continuum limit. In Appendix B, where methods of solution are described, the theoretical background of the direct simulation Monte Carlo method (DSMC method) is explained in a way that can be read by nonmathematicians.

The existence of ghost effects in a gas in the continuum limit makes molecular gas dynamics indispensable to the study of a gas in the continuum limit, which is traditionally discussed by classical fluid dynamics. Ghost and non-Navier–Stokes effects present themselves in well-known classical fluid dynamic problems, such as the Bénard and Taylor–Couette problems; they are discussed in Chapter 8. Another type of ghost effect, the recently proposed infinitesimal curvature effect, is discussed in Chapter 9, where bifurcation of the plane Couette flow, a long-standing problem, is worked out as an example. The discussion on ghost

effects will be essential to a modern treatment of traditional fluid dynamics.

Basic theory is developed in a systematic way and presented in a form easily applicable to practical use. Fundamental examples showing kinetic effects and various interesting physical phenomena are discussed analytically, numerically, or experimentally. Mathematical discussion is on the level of classical advanced calculus; definitions, assumptions, and formulations are stated explicitly. Thus, engineers can apply theoretical works to practical problems, and mathematicians will have access to physically interesting mathematical problems without much difficulty. Readers should be aware of the relationship of the present book to the author's previous one, *Kinetic Theory and Fluid Dynamics* (Birkhäuser, 2002). The latter is a monograph mainly discussing the time-independent problems in Chapter 3 of the present book in more detail. Some supplementary discussions on the subject, including a brief but systematic discussion of its time-dependent problems, are naturally made in the present book. Thus, the two books are complementary. Misprints that are found in the two above-mentioned books will be posted at <http://fd.kuaero.kyoto-u.ac.jp/members/sone>.

The author owes a great deal to many people. He was influenced by fruitful discussion with the late Harold Grad, who offered the author a chance to work with him at the Courant Institute for two years. Collaboration with French mathematicians, especially C. Bardos and F. Golse, was initiated by H. Cabannes's invitation of the author to be a visiting professor at the Université Pierre et Marie Curie. The author enjoyed very fruitful discussions with Tai-Ping Liu and Shih-Hsien Yu, who offered the author their unpublished works. He also enjoyed discussions with L. Arkeryd in the comfortable climate of several Swedish summers. The conversations and correspondences with J. B. Keller and A. Acrivos were instructive. The discussions and collaboration enhanced the content of the book. G. Bird and W. Wagner read Section B.1 on the DSMC method and gave the author useful comments. T. Yano, M. Hasegawa, T. Doi, H. Sugimoto, S. Takata, T. Kataoka, and M. Handa, each kindly examined considerable parts of the draft manuscript carefully, providing helpful suggestions and improvements; Kataoka read more than ninety percent of the manuscript in detail, and Doi examined the whole manuscript with a particular emphasis on typographical errors. T. Doi, H. Sugimoto, S. Takata, T. Kataoka, K. Sawada, and M. Handa helped the author to prepare figures; Sugimoto also worked to modify all of the figure files to the publisher's standard. Tom Grasso's editorial group at Birkhäuser processed the manuscript carefully and efficiently. The author would like to express his thanks for the various courtesies provided and contributions made. After retirement from Kyoto University in 2000, the author has spent most of his time at home writing contributed papers on molecular gas dynamics and two books, the above-mentioned book and the present one. His life during this period was regulated by the regular request of his family dog, Mill, to go out for walks, which must be considered a nonnegligible contribution.

Yoshio Sone  
Kyoto, 2006

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