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Variational Methods in Shape Optimization Problems

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Preface

The fascinating field of shape optimization problems has received a lot of attention in recent years, particularly in relation to a number of applications in physics and engineering that require a focus on shapes instead of parameters or functions. The goal of these applications is to deform and modify the admissible shapes in order to comply with a given cost function that needs to be optimized. In this respect the problems are both classical (as the isoperimetric problem and the Newton problem of the ideal aerodynamical shape show) and modern (reflecting the many results obtained in the last few decades).

The intriguing feature is that the competing objects are *shapes*, i.e., domains of $\mathbb{R}^N$, instead of functions, as it usually occurs in problems of the calculus of variations. This constraint often produces additional difficulties that lead to a lack of existence of a solution and to the introduction of suitable *relaxed* formulations of the problem. However, in certain limited cases an optimal solution exists, due to the special form of the cost functional and to the geometrical restrictions on the class of competing domains.

This volume started as a collection of the lecture notes from two courses given in the academic year 2000–2001 by the authors at the Dipartimento di Matematica Universit di Pisa and at Scuola Normale Superiore di Pisa respectively. The courses were mainly addressed to Ph.D. students and required as background the topics in functional analysis that are typically covered in undergraduate courses. Subsequently, more material has been added to the original base of lecture notes. However, the style of the work remains quite informal and follows, in large part, the lectures as given.

We decided to open the volume by presenting in Chapter 1 some relevant examples of shape optimization problems: the isoperimetric problem, the Newton problem of optimal aerodynamical profiles, the optimal distribution of two different media in a given region, and the optimal shape of a thin insulator around a given conductor. In Chapter 2 we consider the important case where the additional constraint of convexity is assumed on the competing domains: this situation often provides the extra compactness necessary to prove the existence of an optimal shape. A prototype for this class is the Newton problem, where the convexity of the competing bodies permits the existence of an optimal shape, together with some necessary conditions of optimality.

Many shape optimization problems can be seen in the larger framework of optimal control problems: indeed an admissible shape plays the role of an admissible control, and the corresponding state variable is usually the solution of a partial differential equation on the control domain. This point of view is developed in large generality in Chapter 3, together with the corresponding relaxation theory, which provides a general way to construct relaxed solutions through Γ-convergence methods. In Chapter 4 we study variational problems where the Dirichlet region is seen as one of the unknowns, and the corresponding optimization problems are considered. It must be pointed out that, due to the nature of the problem, in general an optimal Dirichlet region does not exist, and a relaxed formulation is needed to better understand the behavior of minimizing sequences.

Contrarily in Chapter 5 we present some particular cases where, due to the presence of suitable geometrical constraints and the monotonicity of the cost functional, a classical unrelaxed optimal solution does exist, admitting a solution in the family of classical admissible domains. Some relevant examples of problems that fulfill the required assumptions are also shown. Chapter 6 deals with the very special case of cost functionals that depend on the eigenvalues of an elliptic operator with Dirichlet conditions on the free boundary; we collected some classical and modern results together with several problems that are still open. Finally, we devote Chapter 7 to the case of shape optimization problems governed by elliptic equations with Neumann conditions on the free boundary. In this case several additional difficulties arise precluding the development of a complete theory; however, we made an effort to treat completely at least the so called problem of optimal cutting, where the existence of an optimal cut can be deduced in full generality.

The work also contains a substantial, yet hardly exhaustive, bibliography. The compilation of a more complete list of references would be prohibitive due to the rapid development of the field and the tremendous volume of associated papers that are regularly published on the subject.

This study can serve as an excellent text for a graduate course in variational methods for shape optimization problems, appealing to both students and instructors alike.

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