Groups, Matrices, and Vector Spaces

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A Group Theoretic Approach to Linear Algebra

Springer

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Foreword

This book is an introduction to group theory and linear algebra from a geometric viewpoint. It is intended for motivated students who want a solid foundation in both subjects and are curious about the geometric aspects of group theory that cannot be appreciated without linear algebra. Linear algebra and group theory are connected in very pretty ways, and so it seems that presenting them together is an appropriate goal. Group theory, founded by Galois to study the symmetries of roots of polynomial equations, was extended by many nineteenth-century mathematicians who were also leading figures in the development of linear algebra such as Cauchy, Cayley, Schur, and Lagrange. It is amazing that such a simple concept has touched so many rich areas of current research: algebraic geometry, number theory, invariant theory, representation theory, combinatorics, and cryptography, to name some. Matrix groups, which are part matrix theory, part linear algebra, and part group theory, have turned out to be richest source of finite simple groups and the basis for the theory of linear algebraic groups and for representation theory, two very active areas of current research that have linear algebra as their basis. The orthogonal and unitary groups are matrix groups that are fundamental tools for particle physicists and for quantum mechanics. And to bring linear algebra in, we should note that every student of physics also needs to know about eigentheory and Jordan canonical form.

For the curious reader, let me give a brief description of what is covered. After a brief preliminary chapter on combinatorics, mappings, binary operations, and relations, the first chapter covers the basics of group theory (cyclic groups, permutation groups, Lagrange's theorem, cosets, normal subgroups, homomorphisms, and quotient groups) and gives an introduction to the notion of a field. We define the basic fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, and discuss the geometry of the complex plane. We also state the fundamental theorem of algebra and define algebraically closed fields. We then construct the prime fields $\mathbb{F}_p$ for all primes p and define Galois fields. It is especially nice to have finite fields, since computations involving matrices over $\mathbb{F}_2$ are delightfully easy. The lovely subject of linear coding theory, which requires $\mathbb{F}_2$, will be treated in due course. Finally, we define polynomial rings and prove the multiple root test.

We next turn to matrix theory, studying matrices over an arbitrary field. The standard results on Gaussian elimination are proven, and *LPDU* factorization is studied. We show that the reduced row echelon form of a matrix is unique, thereby enabling us to give a rigorous treatment of the rank of a matrix. (The uniqueness of the reduced row echelon form is a result that most linear algebra books curiously ignore.) After treating matrix inverses, we define matrix groups, and give examples such as the general linear group, the orthogonal group, and the $n \times n$ permutation matrices, which we show are isomorphic to the symmetric group $S(n)$. We conclude the chapter with the Birkhoff decomposition of the general linear group.

The next chapter treats the determinant. After defining the signature of a permutation and showing that it is a homomorphism, we define $\det(A)$ via the alternating sum over the symmetric group known as Leibniz's formula. The proofs of the product formula (that is, that det is a homomorphism) and the other basic results about the determinant are surprisingly simple consequences of the definition. The determinant is an important and rich topic, so we treat the standard applications such as the Laplace expansion and Cramer's rule, and we introduce the important special linear group. Finally, we consider a recent application of the determinant known as Dodgson condensation.

In the next chapter, finite-dimensional vector spaces, bases, and dimension are covered in succession, followed by more advanced topics such as direct sums, quotient spaces, and the Grassmann intersection formula. Inner product spaces over $\mathbb{R}$ and $\mathbb{C}$ are covered, and in the appendix, we give an introduction to linear coding theory and error-correcting codes, ending with perfect codes and the hat game, in which a player must guess the color of her hat based on the colors of the hats her teammates are wearing.

The next chapter moves us on to linear mappings. The basic properties such as the rank–nullity theorem are covered. We treat orthogonal linear mappings and the orthogonal and unitary groups, and we classify the finite subgroups of $SO(2, \mathbb{R})$. Using the $O(2, \mathbb{R})$ dichotomy, we also obtain Leonardo da Vinci's classification that all finite subgroups of $O(2, \mathbb{R})$ are cyclic or dihedral. This chapter also covers dual spaces, coordinates, and the change of basis formulas for matrices of linear mappings.

We then take up eigentheory: eigenvalues and eigenvectors, the characteristic polynomial of a linear mapping, and its matrix and diagonalization. We show how the Fibonacci sequence is obtained from the eigenvalue analysis of a certain dynamical system. Next, we consider eigenspace decompositions and prove that a linear mapping is semisimple—equivalently, that its matrix is diagonalizable—if and only if its minimal polynomial has simple roots. We give a geometric proof of the principal axis theorem for both Hermitian and real symmetric matrices and for self-adjoint linear mappings. Our proof of the Cayley–Hamilton theorem uses a simple inductive argument noticed by the author and Jochen Kuttler. Finally, returning to the geometry of $\mathbb{R}^3$, we show that $SO(3, \mathbb{R})$ is the group of rotations of $\mathbb{R}^3$ and conclude with the computation of the rotation groups of several of the Platonic solids.

Following eigentheory, we cover the normal matrix theorem and quadratic forms, including diagonalization and Sylvester's law of inertia. Then we classify linear mappings, proving the Jordan–Chevalley decomposition theorem and the existence of the Jordan canonical form for matrices over an algebraically closed field. The final two chapters concentrate on group theory. The penultimate chapter establishes the basic theorems of abstract group theory up to the Jordan-Schreier theorem and gives a treatment of finite group theory (e.g., Cauchy's theorem and the Sylow theorems) using the very efficient approach via group actions and the orbit-stabilizer theorem. We also classify the finite subgroups of $SO(3, \mathbb{R})$. The appendix to this chapter contains a description of how Polish mathematicians reconstructed the German Enigma machine before the Second World War via group theory. This was a milestone in abstract algebra and to this day is surely the most significant application of group theory ever made.

The final chapter is an informal introduction to the theory of linear algebraic groups. We give the basic definitions and discuss the basic concepts: maximal tori, the Weyl group, Borel subgroups, and the Bruhat decomposition. While these concepts were already introduced for the general linear group, the general notions came into use relatively recently. We also consider reductive groups and invariant theory, which are two topics of contemporary research involving both linear algebra and group theory.

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