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Anatoli Andrianov

Introduction to Siegel Modular Forms and Dirichlet Series

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Anatoli Andrianov
Russian Academy of Sciences
Steklov Institute of Mathematics
Fontanka 27
191023 St. Petersburg
Russia
anandr@pdmi.ras.ru

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To Goro Shimura and to my granddaughter Sasha

Preface

Several years ago I was invited to an American university to give one-term graduate course on Siegel modular forms, Hecke operators, and related zeta functions. The idea to present in a concise but basically complete and self-contained form an introduction to an important and developing area based partly on my own work attracted me. I accepted the invitation and started to prepare the course. Unfortunately, the visit was not realized. But the idea of such a course continued to be alive till after a number of years this book was finally completed. I hope that this short book will serve to attract young researchers to this beautiful field, and that it will simplify and make more pleasant the initial steps.

No special knowledge is presupposed for reading this book beyond standard courses in algebra and calculus (one and several variables), although some skill in working with mathematical texts would be helpful. The reader will judge whether the result was worth the effort.

Dedications. The ideas of Goro Shimura exerted a deep influence on the number theory of the second half of the twentieth century in general and on the author's formation in particular.

When Andrè Weil was signing a copy of his "Basic Number Theory" to my son, he wrote in Russian, "To Fedor Anatolievich hoping that he will become a number theoretist". Fedor has chosen computer science. Now I pass on the idea to Fedor's daughter, Alexandra Fedorovna.

Contents. The main objective of this book is to give a concise but basically complete and self-contained introduction to the multiplicative theory of Siegel modular forms, Hecke operators, and zeta functions, including the classical case of modular forms in one variable. Chapter 1 contains a compressed exposition of essential features of the theory of Siegel modular forms of integral weight for congruence subgroups of the symplectic modular group $\mathrm{Sp}_n(\mathbb{Z})$ of arbitrary genus n. Chapter 2 treats analytical properties of radial Dirichlet series attached to modular forms of genera 1 and 2. Chapter 3 is dealing with the abstract theory of Hecke–Shimura rings for symplectic and related group. Action of Hecke operators on Siegel modular forms is considered in Chapter 4. In Chapter 5, we examine applications of

Hecke operators to a study of multiplicative properties of Fourier coefficients of modular forms and the related Euler product factorization of radial Dirichlet series attached to eigenfunctions. This leads us to Hecke zeta functions of modular forms in one variable and to spinor (or Andrianov) zeta functions of Siegel modular forms of genus two. At the end of this chapter we arrive at the proof of the analytic continuation and functional equation (under certain assumptions) of Euler products associated with modular forms of genus two.

The book contains a number of exercises that usually consider some interesting points not included in the main text, partly for the reasons of space and partly because of their special character.

References. I try to present the proofs in full detail whenever it is reasonable and possible. The main text contains no references. Essential references are included in the **Notes** at the end of the book. It should be noted that the author does not pretend to give an encyclopedic survey of modular forms or a complete bibliography, but rather to hint at certain principal points of the theory and illustrate how they have been reached.

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St Petersburg *Anatoli Andrianov*
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Contents

General Notation

The letters $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are reserved for the set of positive rational integers, the set of positive rational prime numbers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

$K[x_1,\ldots,x_n]$ is the ring of polynomials in $x_1,\ldots,x_n$ with coefficients in K, and $\mathbb{A}_n^m$ is the set of all $m \times n-$matrices with entries in a set $\mathbb{A}$.

If M is a matrix, ${}^tM$ always denotes the transpose of M, $\sigma(M)$ for a square M is the trace of M, and $\overline{M}$ for a complex matrix M means the matrix with conjugate entries. If Y is a real symmetric matrix, then $Y > 0$ (resp., $Y \geqq 0$) means that Y is positive definite (resp., positive semidefinite). For two matrices A and B of suitable dimensions we write

$$A[B] = {}^tBAB.$$

Introduction: The Two Features of Arithmetic Zeta Functions

A zeta function in arithmetic is, generally speaking, a generating function for an arithmetic problem written in the form of a Dirichlet series. A well stated zeta function must have at least two principal features: an *Euler product factorization* and an *analytic continuation* over the whole complex plane satisfying *functional equations*. The first reflects relations between the global arithmetic problem and its localizations, while the second provides a kind of reciprocity between the localizations.

Let us illustrate this with some examples

Riemann Zeta Function. The *Riemann zeta function*,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}\, s > 1),$$

is the generating function for the numbers of ideals of given norm in the ring $\mathbb{Z}$ of rational integers. It has an Euler product factorization of the form

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\mathrm{Re}\, s > 1),$$

the product being taken over all rational prime numbers p.

It was Bernhard Riemann who proved in the middle of the nineteenth century that $\zeta(s)$ has an analytic continuation over the whole complex s-plane, is holomorphic except for a simple pole of residue 1 at $s = 1$, and satisfies the functional equation that the function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where Γ is the gamma function, is invariant under the substitution $s \mapsto 1 - s$. He also discovered that the problem of *distribution of prime numbers* is closely connected with the *location of complex zeros* of the zeta function in the vertical strip $0 \leq \Re s \leq 1$. At the end of the century, J. Hádamard and Ch. de la Vallée Poussin proved that $\zeta(s)$ has no zeros on the line $\Re s = 1$, which implied the famous asymptotic formula for the number $\pi(x)$ of prime numbers not exceeding x,

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty).$$

Zeta Function of Algebraic Varieties. The *global zeta function of a nonsingular algebraic variety* V over the field $\mathbb{Q}$ of rational numbers is defined by an Euler product

$$\zeta^*(V,s) = \prod_p \zeta\left(V_p, p^{-s}\right)$$

of *local zeta functions* $\zeta(V_p, p^{-s})$, where p runs over all prime numbers such that V has a "good" and, in particular, nonsingular *reduction* V_p *modulo* p, i.e., the variety over the finite field $\mathbb{F}(p)$ of p elements obtained by replacing equations defining V with corresponding congruences modulo p. The local zeta function is the zeta function of V_p defined by

$$\zeta(V_p, t) = \exp\left(\sum_{\delta=1}^{\infty} N(p^\delta) t^\delta / \delta\right),$$

where $N(p^\delta)$ is the number of points on V_p with coordinates in the finite field $\mathbb{F}(p^\delta)$ of p^δ elements. According to Bernhard Dwork, the local zeta functions $\zeta(V_p, t)$ of nonsingular varieties V_p are rational fractions in t. It follows that the global zeta function $\zeta^*(V,s)$ can be written as a Dirichlet series convergent in a right half-plane of the complex variable s. It is generally believed that the zeta function can be analytically continued over the whole $s-$plane as a meromorphic function and satisfies functional equations, but it is doubtful that a human being living now will see a complete proof.

Nevertheless, even particular cases present considerable interest (all genuine number theory consists of particular cases; everything else is algebra). Let us consider a (*projective*) *elliptic curve*

$$E: \quad y^2 z = x^3 + axz^2 + bz^3 \qquad (a, b \in \mathbb{Z}).$$

The points on E with coordinates in $\mathbb{Q}$ form an abelian group, which we denote by $E_\mathbb{Q}$; a theorem of L.J. Mordell tells us that the group $E_\mathbb{Q}$ is finitely generated, i.e., is a product of a finite group by a lattice of finite rank g. A principal problem of the theory is to determine the group $E_\mathbb{Q}$, and, in particular, to determine the rank g. In the mid 1960's B.J. Birch and H.P.F. Swinnerton-Dyer put forward revolutionary conjectures connecting the group $E_\mathbb{Q}$ with the zeta function $\zeta^*(E,s)$ of the curve. Let us recall some details.

A prime number p is said to be good if it does not divide $6(27b^2 + 4a^3)$. For such a prime p, the reduction

$$E_p: \quad y^2 z \equiv x^3 + axz^2 + bz^3 \pmod{p}$$

of E modulo p is an elliptic curve over $\mathbb{F}(p)$. It is well known that the zeta function of E_p over $\mathbb{F}(p)$ has the form

$$\zeta(E_p, t) = \frac{1 - (1 + p - N(p))t + pt^2}{(1-t)(1-pt)}.$$

Then we may define a zeta function of E by

$$\zeta^*(E,s) = \prod_p \left(1 - (1+p-N(p))p^{-s} + p^{1-2s}\right)^{-1},$$

where the product is taken over all good primes. It converges for $\Re s > 3/2$. Then the main *Birch–Swinnerton-Dyer conjecture* says that $\zeta^*(E,s)$ has a zero of order g at $s=1$. Generally, it is still open.

Zeta Functions of Automorphic Forms. Despite the clear importance of zeta functions of algebraic varieties, algebraic geometry provides no means for their investigation. The only hope is to relate them to techniques coming from an analytic background, probably with *zeta functions of automorphic forms*, which, in contrast, usually have vast means for analytic investigation, but often lack clear arithmetic motivation.

Let us consider the simplest case of a Dirichlet series of modular forms of integral weight for congruence subgroups K of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Let us recall the corresponding definitions.

A function F on the upper half-plane $\mathbb{H} = \{x+iy \in \mathbb{C} \mid y > 0\}$ is said to be a *modular cusp form* of weight k for the group K if it is holomorphic on $\mathbb{H}$, equals zero at all cusps of K, and satisfies

$$(cz+d)^{-k} F\left(\frac{az+b}{ac+d}\right) = F(z) \quad \text{for each} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K.$$

All such functions form a finite dimensional space $\mathfrak{N} = \mathfrak{N}_k(K)$ over the field $\mathbb{C}$ of complex numbers. If the group K contains the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, then every $F \in \mathfrak{N}$ can be presented by a *Fourier series* absolutely convergent on $\mathbb{H}$ of the form

$$F(z) = \sum_{m=1}^{\infty} f(m) e^{2\pi i m z}$$

with constant Fourier coefficients $f(m)$. Let us associate with F the *Dirichlet series of F* defined by

$$Z(F,s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s}.$$

The series converges absolutely in a right half-plane of the variable s, and can be presented there by means of a Mellin integral

$$\Phi(s) = \Phi(F,s) = \int_0^{\infty} F(iy) y^{s-1} dy = (2\pi)^{-s} \Gamma(s) Z(F,s).$$

Suppose that the group K satisfies

$$\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}^{-1} K \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = K$$

for a positive integer q. Then it is easy to check that for each cusp form $F \in \mathfrak{N}$, the function

$$(F \mid \omega)(z) = q^{-k/2} z^{-k} F(-1/qz)$$

again belongs to $\mathfrak{N}$ and satisfies $F \mid \omega \mid \omega = (-1)^k F$. It follows that one can write the direct sum decomposition

$$\mathfrak{N} = \mathfrak{N}^+ + \mathfrak{N}^-, \quad \text{where for} \quad F \in \mathfrak{N}^\pm, \quad F \mid \omega = \pm i^k F.$$

Exercise. Prove the above assertions.

If $F \in \mathfrak{N}^\pm$, then

$$F(i/qy) = \pm(-1)^k q^{k/2} y^k F(iy) \qquad (y > 0),$$

and we can write, for $\Re s$ sufficiently large, that

$$\begin{aligned}
\Phi(s) &= \int_0^{q^{-1/2}} F(iy) y^{s-1} dy + \int_{q^{-1/2}}^{\infty} F(iy) y^{s-1} dy \\
&= \int_{q^{-1/2}}^{\infty} F(i/qy)(1/qy)^{s-1}(1/qy^2) dy + \int_{q^{-1/2}}^{\infty} F(iy) y^{s-1} dy \\
&= \pm(-1)^k q^{k/2-s} \int_{q^{-1/2}}^{\infty} F(iy) y^{k-s-1} dy + \int_{q^{-1/2}}^{\infty} F(iy) y^{s-1} dy.
\end{aligned}$$

Both of the last integrals are holomorphic for all s, and so is the function $\Phi(s)$. Moreover, the last expression implies that

$$\Phi(k-s) = \pm(-1)^k q^{s-k/2} \Phi(s),$$

which is the functional equation for the Dirichlet series $Z(F, s)$.

Note that the simplest of the groups K satisfying the given conditions is the group

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{q} \right\}.$$

The problem of Euler product factorization of the Dirichlet series corresponding to modular forms of integral weight for the groups of the type $\Gamma_0(q)$ was essentially solved by Erich Hecke in 1937 and completed by A.O.L. Atkin and J. Lehner in 1970. In particular, it was found that although the Dirichlet series of a cusp form does not necessarily have an Euler product factorization, the space of cusp forms has a basis consisting of forms with Dirichlet series decomposable into Euler products, which we call the *zeta functions of modular forms*. Such forms can be characterized as eigenfunctions of certain rings of linear operators, the Hecke operators, acting on the spaces $\mathfrak{N}_k(K)$.

Since the nineteenth century, the main arithmetic application of modular forms had been the analytical theory of integral quadratic forms. The reason is that the generating Fourier series with coefficients equal to numbers of integral representations of positive integers by a positive definite integral quadratic form is a modular (not cusp) form. But in the middle of the twentieth century Goro Shimura and Yutaka Taniyama proposed famous conjectures relating modular forms and elliptic curves over $\mathbb{Q}$. The *Shimura–Taniyama conjecture* includes the conjecture that the zeta function $\zeta^*(E,s)$ of every elliptic curve E over $\mathbb{Q}$ completed by appropriate p-factors for bad primes is, in fact, the zeta function of a cusp form of weight 2 for the group $\Gamma_0(q)$, where q is the product of some degrees of bad primes. In 1985, G. Frey made the remarkable observation that this conjecture would imply Fermat's last theorem. The precise relation of the two was established later by K.A. Ribet, which allowed A. Wiles in 1995 to prove the Fermat's last theorem, one of the brightest achievements of mathematics of the twentieth century.

One can hardly doubt that the relation between zeta functions of elliptic curves and zeta functions of modular forms in one variable described by the Shimura–Taniyama conjecture is only a particular case of some general links between global zeta functions of algebraic varieties and zeta functions of automorphic forms. Speaking of abelian varieties in place of elliptic curves, one can expect that modular forms in one variable should be replaced by Siegel modular forms for congruence subgroups of the symplectic modular group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$, and this expectation is supported by some numerical evidence.