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## **PATH PLAYER GAMES**

# Springer Optimization and Its Applications

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## *Aims and Scope*

Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics and other sciences.

*The Springer Series in Optimization and Its Applications* publishes undergraduate and graduate textbooks, monographs and state-of-the-art expository works that focus on algorithms for solving optimization problems and also study applications involving such problems. Some of the topics covered include nonlinear optimization (convex and nonconvex), network flow problems, stochastic optimization, optimal control, discrete optimization, multi-objective programming, description of software packages, approximation techniques and heuristic approaches.

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# PATH PLAYER GAMES

Analysis and Applications

By

SILVIA SCHWARZE

Department of Business and Economics  
University of Hamburg, Germany

 Springer

Silvia Schwarze  
Department of Business and Economics  
University of Hamburg, Germany  
schwarze@econ.uni-hamburg.de

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*To Torsten,  
I am glad that we play on the  
same path.*

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# Contents

<b>Symbols and Abbreviations</b> .....	ix
<b>1 Introduction</b> .....	1
1.1 Network Games .....	1
1.2 The Scope of This Book .....	2
1.3 Acknowledgments .....	5
<b>2 The Path Player Game</b> .....	7
2.1 The Model .....	7
2.1.1 Introduction .....	7
2.1.2 Notation .....	10
2.1.3 The Rules of the Game .....	11
2.1.4 Game Types .....	12
2.1.5 Topology of Networks in Path Player Games .....	15
2.2 Equilibria in Path Player Games .....	19
2.2.1 Introduction .....	19
2.2.2 The One-Dimensional Benefit Function .....	19
2.2.3 Equilibria for General Benefits and Existence of Equilibria .....	22
2.2.4 Special Instances of Path Player Games .....	29
2.2.5 Equilibria for Special Cost Functions .....	35
2.3 Dominated Equilibria .....	49
2.3.1 Introduction .....	49
2.3.2 Relations Between Equilibria and Nondominated Flows .....	52
2.4 Potential Functions for Path Player Games .....	58
2.4.1 Introduction .....	58
2.4.2 Exact Restricted Potential for the Generalized PPG ...	64
2.4.3 An Ordinal Potential Function for Path Player Games .	74
2.4.4 An Exact Potential for an Extended Benefit Function ..	77
2.4.5 Computation of Equilibria by Improvement Sequences .	78

<b>3</b>	<b>Games on Polyhedra: A Generalization</b> .....	85
3.1	Introduction .....	85
3.2	Equilibria and Nondominated Solutions .....	90
3.2.1	Equilibria for Linear Payoffs .....	91
3.2.2	Equilibria for Strictly Increasing Payoffs .....	95
3.2.3	Equilibria for Convex Payoffs .....	97
3.2.4	Nondominated Equilibria .....	98
3.3	Extension to a Game on the Hypercuboid .....	100
3.4	Potentials for Games on Polyhedra .....	104
3.4.1	Potentials in the Extension to the Hypercuboid .....	104
3.4.2	Restricted Potentials in the Original Game .....	111
3.4.3	Computation of Equilibria by Improvement Sequences .....	114
<b>4</b>	<b>The Line Planning Game: An Application</b> .....	117
4.1	Introduction .....	117
4.2	The Line Planning Game Model .....	118
4.3	The PPG as an Instance of the LPG .....	123
4.4	Generalized LPG as an Instance of Games on Polyhedra .....	128
4.4.1	Formulation as a Game on a Polyhedron .....	128
4.4.2	Results of Using Polyhedral Representation .....	130
4.5	Extensions of the Line Planning Game .....	135
4.5.1	Integer Line Planning Game .....	135
4.5.2	Multiple Origin–Destination Pairs .....	138
4.6	Line Planning for Interregional Trains in Germany .....	140
<b>5</b>	<b>Summary</b> .....	155
<b>Appendix</b> .....		159
A.1	Lemma: Transformation of Line Planning Game .....	159
A.2	Polyhedron for Line Planning Game .....	162
<b>References</b> .....		163
<b>Index</b> .....		171

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## Symbols and Abbreviations

### General Symbols

$\mathbf{0}_n$  .....  $n$ -dimensional zero vector  
 $\mathbf{1}_n$  .....  $n$ -dimensional one vector  
 $\mathbb{I}_n$  .....  $n \times n$  identity matrix  
 $\mathbb{N}_0$  .. natural numbers including zero  
 $\mathbb{R}_+$  ..... nonnegative real numbers  
 $\mathbb{R}^n$  ..  $n$ -dimensional Euclidian space  
 $\mathbb{R}_+^n$  ..... nonnegative  $n$ -dimensional  
   Euclidian space  
 $\text{sgn}(a)$  ..... signum of  $a$

### Symbols: Path Player Game

$b(f)$  ..... vector of benefit functions  
 $b_P(f)$  ..... benefit of player  $P$   
 $\tilde{b}_P(f)$  .... one-dimensional benefit of  
   player  $P$   
 $c_e(f_e)$  ..... cost on edge  $e$   
 $c_P(f)$  ..... cost of path  $P$   
 $\tilde{c}_P(f)$  .... one-dimensional cost of  $P$   
 $d_P(f_{-P})$  ..... decision limit  
 $e$  ..... edge  
 $e_m$  ..... class of edges shared by  
   player set  $m$   
 $E$  ..... set of edges  $e$   
 $E_P^{\text{com}}$  ... set of commonly used edges  
 $E_P^{\text{exc}}$  ... set of exclusively used edges  
 $f$  ..... network flow  
 $f_e$  ..... flow on edge  $e$   
 $f_P$  ..... flow on path  $P$

$f_{-P}$  ..... network flow excluding  $P$   
 $f_P^{\text{max}}(f_{-P})$  ... best reaction set of  $P$   
 $\mathbb{F}$  ..... set of feasible flows  
 $\Gamma$  ..... game instance  
 $G$  ..... game network  
 $G(n)$  .standard network of  $n$  players  
 $I(\varphi)$  ..... cost of sequence  $\varphi$   
 $\kappa_P$  ..... security payment of path  $P$   
 $l(\varphi)$  ..... length of sequence  $\varphi$   
 $M$  ... real number, sufficiently large  
 $ND(\Gamma)$  ... set of nondominated flows  
 $NE(\Gamma)$  ..... set of equilibria  
 $\omega_P$  ..... security limit of path  $P$   
 $\Pi(f)$  ..... potential function  
 $\varphi$  ..... strategy sequence  
 $P$  ..... path  
 $\mathcal{P}$  ..... set of paths  $P$   
 $\mathbb{P}(n)$  ..... power set of set of players  
 $r$  ..... flow rate  
 $s$  ..... source vertice  
 $t$  ..... sink vertice  
 $v$  ..... vertice  
 $v_i$  .....  $i$ th unit vector  
 $V$  ..... set of vertices  $v$

### Symbols: Game on Polyhedra

$A$  ..... coefficient matrix  
 $b$  ..... coefficient vector  
 $c_i(x)$  ..... cost of player  $i$



$h_i$ .....	strategy set of player $i$	$\mathbb{F}^{LPG}$ .....	set of feasible frequencies
$H(A, b)$ .....	hypercuboid	$\mathbb{F}^{ILPG}$ .....	set of feasible integer frequencies
$i$ .....	player	$N$ .....	real number, sufficiently large
$n$ .....	number of players	$P$ .....	line
$S(A, b)$ .....	polyhedron of feasible solutions	$Q$ .....	number of OD pairs
$S_i(x_{-i})$	feasible strategies of player $i$	$\mathcal{P}$ .....	line pool
$x$ .....	solution	$\mathcal{P}_q$ .....	line pool of $\{s_q, t_q\}$
$x_i$ .....	strategy of player $i$	$s_q$ .....	$q$ th origin
$x_{-i}$ .....	solution excluding $x_i$	$t_q$ .....	$q$ th destination
$z_m(x_m)$ .....	cost function of set $m$		

*Symbols: Line Planning Game*

$b_P(f)$ .....	payoff of line $P$
$c_P(f)$ .....	cost of line $P$
$d_P^1(f_{-P})$ .....	lower decision limit
$d_P^2(f_{-P})$ .....	upper decision limit
$f$ .....	network frequency
$f_e$ .....	frequency on edge $e$
$f_P$ .....	frequency on path $P$
$f_P^{br}$ .....	best reaction set of $P$
$f^{\min}$ .....	minimal frequency
$f_q^{\min}$ ..	minimal frequency of $\{s_q, t_q\}$
$f_e^{\max}$ ..	maximal frequency on edge $e$

*Abbreviations*

AFIP .....	approximate finite improvement property
FBRP .....	finite best-reply property
FIP .....	finite improvement property
GNE .....	generalized equilibria
ILPG .....	integer line planning game
LPG .....	line planning game
NCS .....	noncompensative-security
OD .....	origin-destination
PPG .....	path player game
QVI .....	quasi-variational inequalities