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Susan Barwick · Gary Ebert

Unitals in Projective Planes

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This book is dedicated to

Rey Casse

and in memory of Richard Bruck

Preface

This book is a monograph on unitals embedded in finite projective planes. Unitals are an interesting structure found in square order projective planes, and numerous research articles constructing and discussing these structures have appeared in print. More importantly, there still are many open problems, and this remains a fruitful area for Ph.D. dissertations.

Unitals play an important role in finite geometry as well as in related areas of mathematics. For example, unitals play a parallel role to Baer subplanes when considering extreme values for the size of a blocking set in a square order projective plane (see Section 2.3). Moreover, unitals meet the upper bound for the number of absolute points of any polarity in a square order projective plane (see Section 1.5). From an applications point of view, the linear codes arising from unitals have excellent technical properties (see Section 6.4). The automorphism group of the classical unital $\mathcal{H} = \mathcal{H}(2, q^2)$ is 2-transitive on the points of $\mathcal{H}$, and so unitals are of interest in group theory. In the field of algebraic geometry over finite fields, $\mathcal{H}$ is a maximal curve that contains the largest number of $\mathbb{F}_{q^2}$-rational points with respect to its genus, as established by the Hasse-Weil bound.

The book is a thorough survey of the research literature on embedded unitals, the first time such material has been collected and presented in one place. The intended audience consists of graduate students and mathematicians who want to learn this subject area without reading all the original articles. Moreover, the book should serve as a useful reference for researchers already working in the area. The primary proof techniques used involve linear algebraic arguments, finite field arithmetic, some elementary number theory, and combinatorial enumeration. Some computer results not previously found in the literature are mentioned in the text.

We assume that the reader has some familiarity with projective planes, and cover the necessary preliminary material in Chapter 1. In Chapter 2 we introduce Hermitian curves in $\mathrm{PG}(2, q^2)$ and study their geometric and combinatorial properties. We then define unitals as a generalization of Hermitian curves, and discuss unitals embedded in projective planes as well as unitals treated independently as designs. We note that while most unitals are not embedded in projective planes, the most interesting ones from a structural point of view are the embedded ones. Blocking sets and their relation to unitals are discussed, providing additional motivation for the study of embedded unitals.

Chapter 3 introduces translation planes and covers in detail some projective geometry techniques that are very useful in the study of unitals. Namely, we define the Bruck-Bose representation in $\mathrm{PG}(4, q)$ for a translation plane of dimension at most two over its kernel. This representation is useful in a much broader context than the study of unitals, and the detailed presentation given here should prove to be invaluable for graduate students working in any area of finite geometry. This naturally leads to a discussion of the construction of spreads in $\mathrm{PG}(3, q)$. Finally, the substructure of square order planes, and of Baer subplanes in particular, then leads to the notion of derivation as a means for constructing new projective planes from old ones.

Chapter 4 begins with Buekenhout's two constructions for embedded unitals. The first produces unitals in any translation plane of dimension at most two over its kernel, and the second produces unitals in certain derivable two-dimensional translation planes. It turns out that the second procedure produces only the classical unital (Hermitian curve) in the Desarguesian plane $\mathrm{PG}(2, q^2)$, while the first procedure produces not only the classical unital but also numerous nonclassical unitals in $\mathrm{PG}(2, q^2)$, as long as $q > 2$. The chapter concludes with a careful description of the known unitals embedded in $\mathrm{PG}(2, q^2)$, all of which may be obtained from Buekenhout's first construction. Coordinates are given for these unitals, their structure is investigated, their groups are determined, and the number of nonequivalent unitals so constructed is determined.

Chapter 5 surveys unitals embedded in non-Desarguesian planes, beginning with a full investigation of the known unitals in the Hall plane $\mathrm{Hall}(q^2)$. In particular, we discuss how these unitals are inherited from unitals in $\mathrm{PG}(2, q^2)$ via derivation. Computer results are given for all possible unitals obtained from either of Buekenhout's two constructions when applied to Hall planes of small order. In particular, we note that there is always one unital which naturally arises from the first Buekenhout construction which is not obtained by any of the above derivation results. The chapter concludes with a survey of the known unitals in semifield planes, nearfield planes, Figueroa planes, and Hughes planes. Again some computer results are given for small orders.

In Chapter 6 we investigate various combinatorial properties of unitals and some associated configurations. We survey results concerning the intersection of a unital with a Baer subplane, as well as the intersection of two unitals. We look at spreads and packings of unitals, discuss the construction of inversive planes from certain unitals, investigate arcs contained in the classical unital, and look at the construction of codes from unitals.

In Chapter 7 a comprehensive survey is given of the known geometric characterizations of ovoidal-Buekenhout-Metz and classical unitals. Results concerning how many chords of a unital U in $\mathrm{PG}(2,q^2)$ need to be Baer sublines in order for U to be classical or ovoidal-Buekenhout-Metz are surveyed and proved in detail. Other configurational characterizations are given, including Thas' proof of the longstanding conjecture that a unital U in $\mathrm{PG}(2,q^2)$ with collinear feet from every point in $\mathrm{PG}(2,q^2)\setminus U$ must be classical. Finally, we briefly look at characterizations of unitals using the Bruck-Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$ and the quadratic extension $\mathrm{PG}(4,q^2)$, as well as characterizations using the Bose representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(5,q)$.

The book concludes with Chapter 8, where a number of open problems are presented. Appendix A provides a standard naming system for the unitals arising from either of Buekenhout's two constructions, and includes a discussion of the conflicting names presently found in the literature. Appendix B catalogues and summarizes group theoretic characterizations of classical and ovoidal-Buekenhout-Metz unitals, whose group theoretic proofs are beyond the scope of this book.

The bibliography is a comprehensive list of research articles on unitals embedded in projective planes, and should be a valuable resource to anyone working in the area. A notation index is provided, as well as a conventional index of all terms used in the book.

The authors would like to thank the referees for valuable comments that improved the clarity and quality of the final version of this monograph. In addition thanks go to the editorial staff at Springer-Verlag for their assistance. Thanks to Rey Casse and Richard Bruck (in memoriam) for introducing us to the delights of finite geometry. Finally, we wish to thank our families for their patience and enduring support, without which this project would never have come to fruition.

Susan Barwick
March, 2008 *Gary Ebert*

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