

The Illusion of Linearity

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The Illusion of Linearity

From Analysis to Improvement

 Springer

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Foreword

In a cubic centimeter, there are 1 000 cubic millimeters, in a cubic decimeter 1 000 000, in a cubic meter 1 000 000 000, and so forth. Why on earth is it usually so difficult to teach and to learn such simple facts, and many others of a similar vein? Of course, some questions of this type are more intricate. There is no easy computation showing that a giant ten times as high as a given dwarf weighs about one thousand more. So the nature of the problem is a crucial factor, and the authors of this study are fully aware of that.

This book deals with the illusion of linearity, mainly in the context of enlargement and reduction of figures and solids. An elementary example is when somebody believes that multiplying the side of a square by 2 implies that its area is also multiplied by 2. The book approaches also, to some extent, the context of probabilities. The authors rely essentially on two methods of investigation, namely experiments involving an experimental and a control group of students, and individual interviews on the other. A number of important variables are scrutinized, the most important being:

- drawings made by students themselves versus ready-made drawings, using squared paper or not
- *direct* versus *indirect* measures
- problems stated in the missing-value format, or in the comparison format
- awakening students' consciousness by a preliminary significant question
- degree of authenticity of the situation.

Two of these variables deserve some further explanation.

Direct versus indirect measures. Expressing an area in square meters is an example of a *direct measure*, while relating the area of a surface to the amount of paint required to cover it is an example of an *indirect measure*.

Missing-value or comparison format. As an example, one side of a polygon measures 2 dm and its area is 6 dm^2 . What happens to the area in an enlargement operation where the 2 dm become 8 dm? This is a missing-value format. And if the 2 dm were increased by a factor of 4? This is a comparison format. In short, the givens are three measures in the former case, and two measures and a ratio of measures in the latter.

An impressive result of the study is how deep-rooted the illusion of linearity is and how strongly it resists many variations of the teaching and learning parameters. The principal circumstance in which the illusion is substantially weakened is when the students are asked, instead of drawing or computing, to physically cover the enlarged surface (of the problem) by an appropriate paving. And even in that case, the improvement of the students' awareness does not substantially withstand returning to more academically stated questions.

The authors also tried a series of ten classroom one-hour sessions inspired by the principles of realistic mathematics education: meaningful and attractive problems, small group work, whole-class discussions and, as far as mathematical matters are concerned, a variety of representations and symbols (drawings, tables of functions, graphs, formulas). Even such a more concentrated and well-oriented pedagogical action did not yield entirely satisfactory results: "many students did not develop a deeper understanding of (non)-proportionality."

For such persistence of the illusion of linearity, three main causes are identified. One of them pertains to the way proportionality is often taught, namely when some parts of the curriculum pay an almost exclusive attention to proportionality as compared to non-linear relations, when there is an overuse of missing-value problems and an overemphasis on routine solving processes as compared to meaningful analysis of situations. Indeed, proportionality is more than a four-term relation. There are the classical rule of three, tables of proportionality showing more than four terms, straight-line graphs passing through the origin, the constant slope of such graphs identified with the coefficient of proportionality, etc. Of course, these features are better understood when contrasted with non-proportional (non-linear) relations. If the teaching ignores these meaningful facets of linearity and remains confined to the narrow domain of four-term missing-value questions, and if it does not contrast proportionality with non-proportionality, then the students are likely to remain like short-sighted prisoners in an obscure intellectual cell. As was so convincingly explained by Wertheimer (1945), a perspicacious problem solver in a given domain is

one who knows the landscape familiarly, i.e. not only its various parts, but the ways to circulate amongst them. Perceiving the very structure as a whole is crucial.

Further, if linearity and non-linearity ought to be regularly confronted, doesn't it mean that the real notion at stake is the one of function with its various modalities? As Klein (1939) wrote a long time ago (the quotation remains surprisingly timely after such a long period):

We, who are readily called reformers, want to place the concept of function at the centre of teaching. For it is that mathematical concept of the last 200 years which, wherever mathematical thought is needed, plays a central role.

Another cause of the persistent illusion of proportionality can be found, according to the authors, in some shortcomings of the general geometrical knowledge of the students. However, this second cause is akin to the first one. One refers to the teaching of proportionality, the other to an unsuccessful teaching of geometry in general. But what does it mean that, when solving proportionality questions, the students show some gaps in their general geometrical knowledge? It means that their understanding of proportionality lacks some structural links with significant adjoining geometrical questions. Generalizing this comment, one might say that mathematics is not a juxtaposition of items, it is an integrated culture. What is at stake is the mobility of mind. The authors are aware of that. As a remedy, they propose to displace the emphasis from computing correct numerical answers to building appropriate mathematical models. But what is the substance of such models if not those mathematical notions and properties that faithfully express the structure of the situation on hand?

Let us now leave aside the deficiencies of the teaching system. A third cause of the illusion of linearity is of a more intrinsic nature. It relates to the intuitiveness and simplicity of the linear relation. This deserves some comments. Let us assume that a student received a fully appropriate instruction on proportionality and non-proportionality. She or he might still be seduced by the charms of proportionality. This is an effect of what might be called the inertia of concepts. Proportionality is similar, to some extent, to a paradigm in the sense of Kuhn (1962). When you have an intellectual instrument at your disposal, if this instrument properly solved a lot of previous problems, if it appears simpler and more elegant than others, then you stick to it until further notice. What happens here pertains simultaneously to the pleasant and simple nature of the knowledge and the indolent nature of the human mind. The inertia of the concepts is also illustrated by a striking finding of the study. In fact, when students have been duly trained, on a number of examples, to identify the non-linear

situations, they show a tendency to overuse a non-linear model. Changing ones mind is not so easy, but once this is done... *Le mieux est l'ennemi du bien*¹.

Now, what could be done to avoid such seductions, if not developing the habit of doubting, a critical mind, a constant circumspection in front of any problem, the habit of checking everything? To conclude, may I express how I appreciate the honesty of this study. It brings us a most careful survey of a number of real difficulties more than a wealth of solutions. All the more so that one of the findings is that even ten classroom sessions are not enough to bring a persistent change. No panacea is proposed. These questions have to be considered in a long run perspective. In the mean time, some doubts will remain. But after all, as Dante wrote in the *Inferno*,

Che non men che saper, dubbiar m'aggrada,

which means

As well as knowing, doubting is praiseworthy.

Nicolas Rouche
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¹ The best is the enemy of the good.

Acknowledgments

In this monograph we report and reflect upon a series of studies that have been done over the past years on the 'illusion of linearity'. Although this book is the product of several years of intensive collaboration between the four authors, many other people had played a part in shaping the overall research program, in designing the experiments, in analysing and interpreting the outcomes, and – last but not least – in preparing the manuscript.

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