Undergraduate Texts in Mathematics

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A Course in Calculus and Real Analysis

With 71 Figures

Springer

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Preface

Calculus is one of the triumphs of the human mind. It emerged from investigations into such basic questions as finding areas, lengths and volumes. In the third century B.C., Archimedes determined the area under the arc of a parabola. In the early seventeenth century, Fermat and Descartes studied the problem of finding tangents to curves. But the subject really came to life in the hands of Newton and Leibniz in the late seventeenth century. In particular, they showed that the geometric problems of finding the areas of planar regions and of finding the tangents to plane curves are intimately related to one another. In subsequent decades, the subject developed further through the work of several mathematicians, most notably Euler, Cauchy, Riemann, and Weierstrass.

Today, calculus occupies a central place in mathematics and is an essential component of undergraduate education. It has an immense number of applications both within and outside mathematics. Judged by the sheer variety of the concepts and results it has generated, calculus can be rightly viewed as a fountainhead of ideas and disciplines in mathematics.

Real analysis, often called mathematical analysis or simply analysis, may be regarded as a formidable counterpart of calculus. It is a subject where one revisits notions encountered in calculus, but with greater rigor and sometimes with greater generality. Nonetheless, the basic objects of study remain the same, namely, real-valued functions of one or several real variables.

This book attempts to give a self-contained and rigorous introduction to calculus of functions of one variable. The presentation and sequencing of topics emphasizes the structural development of calculus. At the same time, due importance is given to computational techniques and applications. In the course of our exposition, we highlight the fact that calculus provides a firm foundation to several concepts and results that are generally encountered in high school and accepted on faith. For instance, this book can help students get a clear understanding of (i) the definitions of the logarithmic, exponential and trigonometric functions and a proof of the fact that these are not algebraic functions, (ii) the definition of an angle and (iii) the result that the ratio of the circumference of a circle to its diameter is the same for all circles. It is our experience that a majority of students are unable to absorb these concepts and results without getting into vicious circles. This may partly be due to the division of calculus and real analysis in compartmentalized courses. Calculus is often taught as a service course and as such there is little time to dwell on subtleties and gain perspective. On the other hand, real analysis courses may start at once with metric spaces and devote more time to pathological examples than to consolidating students' knowledge of calculus. A host of topics such as L'Hôpital's rule, points of inflection, convergence criteria for Newton's method, solids of revolution, and quadrature rules, which may have been inadequately covered in calculus courses, become passé when one studies real analysis. Trigonometric, exponential, and logarithmic functions are defined, if at all, in terms of infinite series, thereby missing out on purely algebraic motivations for introducing these functions. The ubiquitous role of π as a ratio of various geometric quantities and as a constant that can be defined independently using calculus is often not well understood. A possible remedy would be to avoid the separation of calculus and real analysis into seemingly disjoint courses and textbooks. Attempts along these lines have been made in the past as in the excellent books of Hardy and of Courant and John. Ours is another attempt to give a unified exposition of calculus and real analysis and address the concerns expressed above. While this book deals with functions of one variable, we intend to treat functions of several variables in another book.

The genesis of this book lies in the notes we prepared for an undergraduate course at the Indian Institute of Technology Bombay in 1997. Encouraged by the feedback from students and colleagues, the notes and problem sets were put together in March 1998 into a booklet that has been in private circulation. Initially, it seemed that it would be relatively easy to convert that booklet into a book. Seven years have passed since then and we now know a little better! While that booklet was certainly helpful, this book has evolved to acquire a form and philosophy of its own and is quite distinct from the original notes.

A glance at the table of contents should give the reader an idea of the topics covered. For the most part, these are standard topics and novelty, if any, lies in how we approach them. Throughout this text we have sought to make a distinction between the intrinsic definition of a geometric notion and the analytic characterizations or criteria that are normally employed in studying it. In many cases we have used articles such as those in *A Century of Calculus* to simplify the treatment. Usually each important result is followed by two kinds of examples: one to illustrate the result and the other to show that a hypothesis cannot be dropped.

When a concept is defined it appears in boldface. Definitions are not numbered but can be located using the index. Everything else (propositions, examples, remarks, etc.) is numbered serially in each chapter. The end of a proof is marked by the symbol □, while the symbol ◇ marks the end of an example or a remark. Bibliographic details about the books and articles mentioned in the text and in this preface can be found in the list of references. Citations within the text appear in square brackets. A list of symbols and abbreviations used in the text appears after the list of references.

The *Notes and Comments* that appear at the end of each chapter are an important part of the book. Distinctive features of the exposition are mentioned here and often pointers to some relevant literature and further developments are provided. We hope that these may inspire many fruitful visits to the library—not when a quiz or the final is around the corner, but perhaps after it is over. The *Notes and Comments* are followed by a fairly large collection of exercises. These are divided into two parts. Exercises in Part A are relatively routine and should be attempted by all students. Part B contains problems that are of a theoretical nature or are particularly challenging. These may be done at leisure. Besides the two sets of exercises in every chapter, there is a separate collection of problems, called Revision Exercises which appear at the end of Chapter 7. It is in Chapter 7 that the logarithmic, exponential, and trigonometric functions are formally introduced. Their use is strictly avoided in the preceding chapters. This meant that standard examples and counterexamples such as $x \sin(1/x)$ could not be discussed earlier. The Revision Exercises provide an opportunity to revisit the material covered in Chapters 1–7 and to work out problems that involve the use of elementary transcendental functions.

The formal prerequisites for this course do not go beyond what is normally covered in high school. No knowledge of trigonometry is assumed and exposure to linear algebra is not taken for granted. However, we do expect some mathematical maturity and an ability to understand and appreciate proofs. This book can be used as a textbook for a serious undergraduate course in calculus. Parts of the book could be useful for advanced undergraduate and graduate courses in real analysis. Further, this book can also be used for self-study by students who wish to consolidate their knowledge of calculus and real analysis. For teachers and researchers this may be a useful reference for topics that are usually not covered in standard texts.

Apart from the first paragraph of this preface, we have not discussed the history of the subject or placed each result in historical context. However, we do not doubt that a knowledge of the historical development of concepts and results is important as well as interesting. Indeed, it can greatly enrich one's understanding and appreciation of the subject. For those interested, we encourage looking on the Internet, where a wealth of information about the history of mathematics and mathematicians can be readily found. Among the various sources available, we particularly recommend the MacTutor History of Mathematics archive `http://www-groups.dcs.st-and.ac.uk/history/` at the University of St. Andrews. The books of Boyer, Edwards, and Stillwell are also excellent sources for the history of mathematics, especially calculus.

In preparing this book we have received generous assistance from various organizations and individuals. First, we thank our parent institution IIT Bombay and in particular its Department of Mathematics for providing excellent infrastructure and granting a sabbatical leave for each of us to work on this book. Financial assistance for the preparation of this book was received from the Curriculum Development Cell at IIT Bombay, for which we are thankful. Several colleagues and students have read parts of this book and have pointed out errors in earlier versions and made a number of useful suggestions. We are indebted to all of them and we mention, in particular, Rafikul Alam, Swanand Khare, Rekha P. Kulkarni, Narayanan Namboodri, S. H. Patil, Tony J. Puthenpurakal, P. Shunmugaraj, and Gopal K. Srinivasan. The figures in the book have been drawn using PSTricks, and this is the work of Habeeb Basha and to a greater extent of Arunkumar Patil. We appreciate their efforts, and are grateful to them. Thanks are also due to C. L. Anthony, who typed a substantial part of the manuscript. The editorial and TeXnical staff at Springer, New York, have always been helpful and we thank all of them, especially Ina Lindemann and Mark Spencer for believing in us and for their patience and cooperation. We are also grateful to David Kramer, who did an excellent job of copyediting and provided sound counsel on linguistic and stylistic matters. We owe more than gratitude to Sharmila Ghorpade and Nirmala Limaye for their support.

We would appreciate receiving comments, suggestions, and corrections. These can be sent by e-mail to `acicara@gmail.com` or by writing to either of us. Corrections, modifications, and relevant information will be posted at `http://www.math.iitb.ac.in/~srg/acicara` and we encourage interested readers to visit this website to learn about updates concerning the book.

Mumbai, India
July 2005

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Balmohan Limaye

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