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Sophia L. Kalpazidou

Cycle Representations of Markov Processes

Second Edition

With 17 Figures

Springer

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To my father

Preface to the Second Edition

The cycle representations of Markov processes have been advanced after the publication of the first edition to many directions. One main purpose of these advances was the revelation of wide-ranging interpretations of the cycle decompositions of Markov processes such as homologic decompositions, orthogonality equations, Fourier series, semigroup equations, disintegrations of measures, and so on, which altogether express a genuine law of real phenomena.

The versatility of these interpretations is consequently motivated by the existence of algebraic–topological principles in the fundamentals of the cycle representations of Markov processes, which eliberates the standard view on the Markovian modelling to new intuitive and constructive approaches. For instance, the ruling role of the cycles to partition the finite-dimensional distributions of certain Markov processes updates Poincare's spirit to describing randomness in terms of the discrete partitions of the dynamical phase state; also, it allows the translation of the famous Minty's painting lemma (1966) in terms of the stochastic entities.

Furthermore, the methods based on the cycle formula of Markov processes are often characterized by minimal descriptions on cycles, which widely express a philosophical analogy to the Kolmogorovean entropic complexity. For instance, a deeper scrutiny on the induced Markov chains into smaller subsets of states provides simpler descriptions on cycles than on the stochastic matrices involved in the "taboo probabilities." Also, the recurrence criteria on cycles improve previous conditions based on the stochastic matrices, and provide plenty of examples.

The second edition unifies all the interpretations and trends of the cycle representations of Markov processes in the following additional chapters:

Chapter 8: Cycloid Markov Processes,
Chapter 9: Markov Processes on Banach Spaces on Cycles,
Chapter 10: The Cycles Measures,
Chapter 11: Wide Ranging Interpretations of the Cycle Representations.

Apart of that, it contains the new section 3.6 of Part I devoted to the induced circuit chains, and the section 1.4 of Part II devoted to "the recurrence criterions in terms of the weighted circuits for unidimensional random walks in random environment."

Also, some improvements are introduced along the lines of the initial edition.

I would like to thank Professor Y. Derriennic for his persevering contributions to the present edition expressed especially by his prototypes on the recurrence of unidimensional random walks in random environment.

Interesting ideas and results to Banach spaces on cycles are due to my collaboration with N. Kassimatis, Ch. Ganatsiou, and Joel E. Cohen.

Also, the Chinese School of Peking (Qian Minping, Qian Gong, Qian Min, Qian Cheng, Gong Guang, Jiang Da-Quan, Guang-Lu Gong, Hong Qian, and others) have been in parallel advanced the cycle representations to interesting applications in biomathematics and physics.

Finally, we all hope that the second edition will further encourage the research on the cycle theory and its impetus to Probability Theory, Measure Theory, Algebraic Topology, Mathematical Analysis, and related fields.

Thessaloniki Sophia L. Kalpazidou

Preface

Ἁρμονίη ἀφανὴς φανερῆς κρέσσων.
Ηράκλειτος

Unrevealed harmony is superior to the visible one.
HERACLITOS

The purpose of the present book is to give a systematic and unified exposition of stochastic processes of the Markovian type, homogeneous and with either discrete or continuous parameter, which, under an additional assumption concerning the existence of invariant measures, can be defined by directed cycles or circuits. These processes are called *cycle* (or *circuit*) *processes*, and the corresponding collections of weighted cycles are called *cycle representations*.

The descriptions of the Markov transition law in terms of the cycles disclose new and special properties which have given an impetus to very intensive research concentrated on the connections between the geometric properties of the trajectories and the algebraic characterization of Markov processes.

Let us start with a few heuristic motivations for this new topic. The simplest example leading to a cycle process arises when modeling the motion of a particle on a closed curve. Observe the particle's motion through $p\,(\geq 1)$ points of this curve at moments one unit of time apart. This amounts to a discretization of the curve into an ordered sequence $c = (c(n), c(n+1), \ldots, c(n+p-1), c(n)), n = 0, \pm 1, \pm 2, \ldots$, called a directed circuit with period $p(c) = p$. The subsequence $\hat{c} = (c(n), c(n+1), \ldots, c(n+p-1))$ will be called a directed cycle (associated with the circuit c). Assign a positive number w_c to c. Then, a normalized measure of the passage from state $i = c(n)$ to $j = c(n+1)$ is given by $w_c/w_c = 1$. Therefore, if no influences occur, the passages from i to j can be codified by an infinite binary sequence $y_{(i,j)} = (0, 1, 0, \ldots, 0, 1, \ldots)$ where 1 or 0 means that at some moment n the particle passes through or does not pass through (i, j).

The sequence $y_{(i,j)}$ is understood as a "nonrandom" sequence in the context of Kolmogorov's theory of complexities since both 1 and 0 appear periodically after each p steps. This happens because of the small complexity of the particle's trajectory which consists of a circuit c alone. Then, when some "chaos" arises, it necessarily presupposes some complexity in the form of the particle's trajectory. So, let us consider a further two, or even more than two, overlapping directed circuits $c_1, \ldots, c_r, r \geq 2$, each associated with some positive number $w_{c_l}, l = 1, \ldots, r$. Imagine that the particle appears sometime at the incident point i of certain circuits, say for simplicity, $c_1, \ldots, c_l, l \leq r$. Then, the particle can continue its motion to another point j through which some circuits $c_{m_1}, \ldots, c_{m_s}, s \leq l, m_1, \ldots, m_s \in \{1, \ldots, l\}$, pass. A natural measure for the particle's transition when moving from i to j can be defined as

$$(w_{c_{m_1}} + w_{c_{m_2}} + \cdots + w_{c_{m_s}})/(w_{c_1} + w_{c_2} + \cdots + w_{c_l})\,. \tag{1}$$

Accordingly, the binary sequence codifying as above the appearances of the pair (i, j) along a trajectory is given by a "more chaotic" sequence like $y_{(i,j)} = (0, 0, 0, 1, 0, 1, 0, 0, 1, 0, \ldots)$, where 1 means that at some moment of time the particle passes through certain circuits containing (i, j). Furthermore, since expression (1) provides transition probability from i to j of a Markov chain $\xi = (\xi_n, n = 0, 1, \ldots)$ we conclude that:

there exist deterministic constructions to a Markov chain ξ which rely on collections $\mathscr{C}$ of directed circuits endowed with certain measures $\mathscr{W} = (w_c, c \in \mathscr{C})$. The pairs $(\mathscr{C}, \mathscr{W})$ completely determine the process ξ.

But the same conclusion can be conversely viewed as:

there are Markov chains ξ which are defined by two distinct entities: a topological entity given by a collection $\mathscr{C}$ of directed circuits, and an algebraic entity given by a measure $\mathscr{W} = (w_c, c \in \mathscr{C})$.

Plainly, both topological and algebraic components $\mathscr{C}$ and $\mathscr{W}$ are not uniquely determined, and this is motivated by the algebraic nature of our construction. To assure the uniqueness, we should look for another approach which can express the definite characteristic of the finite-dimensional distributions of ξ.

A natural way to obtain a uniqueness criterion for $(\mathscr{C}, \mathscr{W})$ can be given by a behavioral approach. It is this approach that we shall further use.

Let S be a finite set, and let $\xi = (\xi_n)_{n\geq 0}$ be a homogeneous and irreducible S-state Markov chain whose transition matrix is $P = (p_{ij}, i, j \in S)$. Denote by $\pi = (\pi_i, i \in S)$ the invariant probability distribution of P, that is, $\pi_i > 0, \sum_i \pi_i = 1$, and

$$\sum_j \pi_i p_{ij} = \sum_j \pi_j p_{ji}, \qquad i \in S. \tag{2}$$

It turns out that system (2) of the "balance equations" can be equivalently written as follows:

$$\pi_i p_{ij} = \sum_c w_c J_c(i,j), \qquad i, j \in S, \tag{3}$$

where c ranges over a collection $\mathscr{C}$ of directed cycles (or circuits) in S, w_c are positive numbers, and $J_c(i,j) = 1$ or 0 according to whether or not (i,j) is an edge of c.

The equivalence of the systems (2) and (3) pressupposes the existence of an invertible transform of the "global coordinates" expressed by the cycle-weights $w_c, c \in \mathscr{C}$, into the "local coordinates" given by the edge-weights $\pi_i p_{ij}, i, j \in S$. That is, geometry (topology) enters into equations (2) and (3). The inverse transform of the edge-coordinates $\pi_i p_{ij}, i, j \in S$, into the cycles ones $w_c, c \in \mathscr{C}$, is given by equations of the form

$$w_c = (\pi_{i_1} p_{i_1 i_2}) \cdots (\pi_{i_{s-1}} P_{i_{s-1} i_s})(\pi_{i_s} p_{i_s i_1})\psi, \tag{4}$$

where $c = (i_1, i_2, \ldots, i_s, i_1), s > 1$, with $i_l \neq i_k, l, k = 1, \ldots, s, l \neq k$, and ψ is a function depending on $i_1, \ldots, i_s, P$, and π. The w_c's have frequently physical counterparts in what are called "through-variables."

To conclude, any irreducible (in general, recurrent) Markov chain ξ admits two equivalent definitions. A first definition is given in terms of a stochastic matrix $P = (p_{ij})$ which in turn provides the edge-coordinates $(\pi_i p_{ij})$, and a second definition is given in terms of the cycle-coordinates $(w_c, c \in \mathscr{C})$.

To see how the edges and cycles describe the random law, we shall examine the definitions of the $\pi_i p_{ij}$ and w_c in the context of Kolmogorov's theory of complexities as exposed in his last work with V.A. Uspensky "Algorithms and Randomness" (see also A.N. Kolmogorov (1963, 1968, 1969, 1983a, b) and V.A. Uspensky and A.L. Semenov (1993)).

Kolmogorov defined the entropy of a binary sequence using the concept of complexity as follows: Given a mode (method) M of description, the complexity $K_M(y_n)$ of any finite string $y_n = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}), n \geq 1$, under the mode M is defined to be the minimal (space) length of a description of y_n in this mode (since there can be several descriptions with respect to M). (We have to note here that there are two types of lengths: the time length and the space length; see A.N. Uspensky and A.L. Semenov (1993), pp. 52–53). Then, considering the class $\mathscr{M}$ of all computable modes of description of a set Y of objects (to which y_n belongs), Kolmogorov proved that there is an optimal mode $\boldsymbol{O}$ of description, not necessarily unique, which provides the shortest possible descriptions, that is, $K_{\boldsymbol{O}}(y_n) \leq K_M(y_n) +$ constant, for all $M \in \mathscr{M}$. The complexity $K_{\boldsymbol{O}}(y_n)$ is called the entropy of y_n.

Now, turning back to our question of how the edges and cycles provide descriptions of the probability distribution $\text{Prob}(\xi_k = i, \xi_{k+1} = j), i, j \in S$, we shall examine the binary sequences assigned to this distribution. To this end let us fix a pair (i, j) of states. Then for any k the probability

$$\begin{aligned}&\text{Prob}(\xi_k = i, \xi_{k+1} = j)\\&\quad = \lim_{n\to\infty} \frac{1}{n}\text{card}\,\{m \leq n-1\colon \xi_m(\omega) = i, \xi_{m+1}(\omega) = j\} \quad \text{a.s.}\end{aligned} \tag{5}$$

can be assigned to an infinite binary sequence $y_{(i,\,j)} = y_{(i,\,j)}(\omega) \equiv (y(0), y(1), \ldots, y(m), \ldots)$ whose coordinates are defined as

$$y(m) = \begin{cases} 1, & \text{if the directed pair}\,(i,j)\,\text{occurs on}\,\omega \text{ at the time } m;\\ 0, & \text{otherwise;}\end{cases} \tag{6}$$

where ω is suitably chosen from the convergence set of (5). A directed pair (i, j) occurs on trajectory ω at moment m if $\xi_{m-1}(\omega) = i$ and $\xi_m(\omega) = j$.

On the other hand, it turns out that the recurrent behavior of ξ determines the appearances of directed circuits $c = (i_1, i_2, \ldots, i_s, i_1), s \geq 1$, with distinct points $i_1, i_2, \ldots, i_s$ when $s > 1$, along the sample paths, whose weights w_c are given by

$$w_c = \lim_{n\to\infty} \frac{1}{n}\,\text{card}\{m \leq n-1\text{: the cycle } \hat{c} \text{ appears, modulo the cyclic permutations, along } \omega\}, \tag{7}$$

almost surely, where m counts the appearances of the cycle $\hat{c}$.

Equations (5) and (7) are connected by the following relation:

$$\begin{aligned}&\frac{1}{n}\,\text{card}\{m \leq n-1\text{: } \xi_m(\omega) = i, \xi_{m+1}(\omega) = j\}\\&\quad = \sum_c \frac{1}{n} w_{c,n}(\omega) J_c(i,j) + \frac{\varepsilon_n(\omega)}{n},\end{aligned} \tag{8}$$

where $\hat{c}$ ranges over the set $\mathscr{C}_n(\omega)$ containing all the directed cycles occurring until n along $\omega, w_{c,n}(\omega)$ denotes the number of the appearances of $\hat{c}$ up to n along ω, and $\varepsilon_n(\omega) = 0$ or 1 according to whether or not the last step from i to j corresponds or does not correspond to an edge of a circuit appearing up to n. Then, we may assign the $y_{(i,j)}(\omega)$ above to another description, say, (0, 0, 0, 0, 1, 0, 0, 1, 0, . . .), where 1 codifies the appearances of a circuit passing through (i, j) along $(\xi_k(\omega))_k$ at certain moments.

Now we shall appeal to Kolmogorov's theory of complexities which, as we have already seen, uses an object-description relation. Accordingly, the object to be considered here will be the binary sequence $y_n = (y(0), y(1), \ldots, y(n-1))$ whose coordinates are defined by (6), while the corresponding descriptions will be expressed in terms of two modes of description as follows.

One mode of description for the y_n will use the edges and will be denoted by E. The corresponding description in the mode E for each finite string $y_n = (y(0), y(1), \ldots, y(n-1))$ is given by the binary sequence $x = (x(0), x(1), \ldots, x(n-1))$ whose coordinates are defined as

$$x(m) = y(m), \qquad m = 0, 1, \ldots, n-1.$$

The second mode of description, denoted by C, is based on the directed cycles, and the corresponding description of y_n above in the mode C is given by the sequence $z = (z(0), \ldots, z(n-1))$ where

$$z(m) = \begin{cases} 1, & \text{if a cycle passing through } (i,j) \text{ occurs along } \omega \text{ at moment } m; \\ 0, & \text{otherwise;} \end{cases}$$

for all $m = 0, 1, \ldots, n-1$.

Nevertheless, it seems that another mode of description would be given by the k-cells, $k = 0, 1, 2$, were we to extend the graph of ξ to the next higher topological structure which is the corresponding 2-complex. But in this case a serious drawback would arise: the descriptions in terms of the k-cells would comprise surface elements (the 2-cells) so that no reasonable algorithmic device would be considered. This motivates the choice of the mode C of description in preference to that provided by the k-cells, $k = 0, 1, 2$, and in this direction we find another two strengthening arguments. First, the replacement of the 2-cells by their bounding circuits leaves invariant the orthogonality equation of the boundary operators which act on the k-cells, $k = 0, 1, 2$; that is, the boundary operators connecting the homology sequence circuits–edges–points will still satisfy the orthogonality equation. Then the use of the 2-cells instead of the circuits becomes superfluous.

Second, the circuit-weights w_c given by (7) enjoy a probabilistic interpretation: *w_c is the mean number of occurrences of c along almost all the sample paths of ξ*. Furthermore, the circuits (cycles) used in mode C can be determined by suitable equations called *cycle generating equations*.

To conclude, the cycles and edges provide two methods of description connected by equation (8). Under this light, cycle representation theory of Markov processes is devoted to the study of the interconnections between the edge-coordinates and cycle-coordinates along with the corresponding implications for the study of the stochastic properties of the processes. Only after the definition of the cycle representations for continuous parameter Markov processes can the idea of separating the geometric (topological) ingredients from their algebraic envelope become clear and lead to the investigations of fine stochastic properties such as Lévy's theorem concerning the positiveness of the transition probabilities.

A systematic development of the fundamentals of the cycle theory, in the spirit of Kolmogorov's algorithmic approach to chaos, started in the 1980s at Thessaloniki from the idea of interconnecting the principles of algebraic topology (network theory), algebra, convex analysis, theory of algorithms, and stochastic processes. For instance, the resulting cycle-decomposition-formula provides the homological dimension of Betti, the algebraic dimension of Carathéodory, and the rotational dimension as new revelations of the Markov processes.

Another school, which developed independently the cycle representations, is that of Qians in Peking (Qian Gong, Qian Minping, Qian Min, Qian Cheng, Gong Guang, Guang-Lu Gong, and others). The Chinese school, using mainly a behavioral approach, defined and explored with exceptional completeness the probabilistic analogues of certain basic concepts which rule nonequilibrium statistical physics such as Hill's cycle flux, Schnakenberg's entropy production, the detailed balance, etc. For instance, conceived as a function on cycles, the entropy production can be regarded as a measure for characterizing how far a process is from being reversible.

In France, Y. Derriennic advanced the cycle representation theory to the study of ergodic problems on random walks in random environment.

Finally, a fourth trend to cycle theory is based on the idea of Joel E. Cohen under the completion of S. Alpern, and this author, for defining a finite recurrent stochastic matrix by a rotation of the circle and a partition whose elements consist of finite unions of the circle-arcs. Recent works of the author have given rise to a theoretical basis, argued by algebraic topology, for developing the rotational idea into an independent setting called the rotational theory of Markov processes. This monograph exposes the results of all the authors who contributed to this theory, in a separate chapter.

The present book is a state-of-the-art survey of all these principal trends to cycle theory, unified in a systematic and updated, but not closed, exposition. The contents are divided into two parts. The first, called "Fundamentals of the Cycle Representations of Markov Processes," deals with the basic concepts and equations of the cycle representations. The second part, called "Applications of the Cycle Representations," is the application of the theory to the study of the stochastic properties of Markov processes.

Sophia L. Kalpazidou

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