

Problem Books in Mathematics

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The IMO Compendium

A Collection of Problems Suggested for the
International Mathematical Olympiads:
1959–2004

With 200 Figures

 Springer

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Preface

The International Mathematical Olympiad (IMO) is nearing its fiftieth anniversary and has already created a very rich legacy and firmly established itself as the most prestigious mathematical competition in which a high-school student could aspire to participate. Apart from the opportunity to tackle interesting and very challenging mathematical problems, the IMO represents a great opportunity for high-school students to see how they measure up against students from the rest of the world. Perhaps even more importantly, it is an opportunity to make friends and socialize with students who have similar interests, possibly even to become acquainted with their future colleagues on this first leg of their journey into the world of professional and scientific mathematics. Above all, however pleasing or disappointing the final score may be, preparing for an IMO and participating in one is an adventure that will undoubtedly linger in one's memory for the rest of one's life. It is to the high-school-aged aspiring mathematician and IMO participant that we devote this entire book.

The goal of this book is to include all problems ever shortlisted for the IMOs in a single volume. Up to this point, only scattered manuscripts traded among different teams have been available, and a number of manuscripts were lost for many years or unavailable to many.

In this book, all manuscripts have been collected into a single compendium of mathematics problems of the kind that usually appear on the IMOs. Therefore, we believe that this book will be the definitive and authoritative source for high-school students preparing for the IMO, and we suspect that it will be of particular benefit in countries lacking adequate preparation literature. A high-school student could spend an enjoyable year going through the numerous problems and novel ideas presented in the solutions and emerge ready to tackle even the most difficult problems on an IMO. In addition, the skill acquired in the process of successfully attacking difficult mathematics problems will prove to be invaluable in a serious and prosperous career in mathematics.

However, we must caution our aspiring IMO participant on the use of this book. Any book of problems, no matter how large, quickly depletes itself if

the reader merely glances at a problem and then five minutes later, having determined that the problem seems unsolvable, glances at the solution.

The authors therefore propose the following plan for working through the book. Each problem is to be attempted at least half an hour before the reader looks at the solution. The reader is strongly encouraged to keep trying to solve the problem without looking at the solution as long as he or she is coming up with fresh ideas and possibilities for solving the problem. Only after all venues seem to have been exhausted is the reader to look at the solution, and then only in order to study it in close detail, carefully noting any previously unseen ideas or methods used. To condense the subject matter of this already very large book, most solutions have been streamlined, omitting obvious derivations and algebraic manipulations. Thus, reading the solutions requires a certain mathematical maturity, and in any case, the solutions, especially in geometry, are intended to be followed through with pencil and paper, the reader filling in all the omitted details. We highly recommend that the reader mark such unsolved problems and return to them in a few months to see whether they can be solved this time without looking at the solutions. We believe this to be the most efficient and systematic way (as with any book of problems) to raise one's level of skill and mathematical maturity.

We now leave our reader with final words of encouragement to persist in this journey even when the difficulties seem insurmountable and a sincere wish to the reader for all mathematical success one can hope to aspire to.

Belgrade,
October 2004

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For the most current information regarding The IMO Compendium you are invited to go to our website: www.imo.org.yu. At this site you can also find, for several of the years, scanned versions of available original shortlist and longlist problems, which should give an illustration of the original state the IMO materials we used were in.

We are aware that this book may still contain errors. If you find any, please notify us at imo@matf.bg.ac.yu. A full list of discovered errors can be found at our website. If you have any questions, comments, or suggestions regarding both our book and our website, please do not hesitate to write to us at the above email address. We would be more than happy to hear from you.

Acknowledgements

The making of this book would have never been possible without the help of numerous individuals, whom we wish to thank.

First and foremost, obtaining manuscripts containing suggestions for IMOs was vital in order for us to provide the most complete listing of problems possible. We obtained manuscripts for many of the years from the former and current IMO team leaders of Yugoslavia / Serbia and Montenegro, who carefully preserved these valuable papers throughout the years. Special thanks are due to Prof. Vladimir Mičić, for some of the oldest manuscripts, and to Prof. Zoran Kadelburg. We also thank Prof. Djordje Dugošija and Prof. Pavle Mladenović. In collecting shortlisted and longlisted problems we were also assisted by Prof. Ioan Tomescu from Romania and Hà Duy Hưng from Vietnam.

A lot of work was invested in cleaning up our giant manuscript of errors. Special thanks in this respect go to David Kramer, our copy-editor, and to Prof. Titu Andreescu and his group for checking, in great detail, the validity of the solutions in this manuscript, and for their proposed corrections and alternative solutions to several problems. We also thank Prof. Abderrahim Ouardini from France for sending us the list of countries of origin for the shortlisted problems of 1998, Prof. Dorin Andrica for helping us compile the list of books for reference, and Prof. Ljubomir Čukić for proofreading part of the manuscript and helping us correct several errors.

We would also like to express our thanks to all anonymous authors of the IMO problems. It is a pity that authors' names are not registered together with their proposed problems. Without them, the IMO would obviously not be what it is today. In many cases, the original solutions of the authors were used, and we duly acknowledge this immense contribution to our book, though once again, we regret that we cannot do this individually. In the same vein, we also thank all the students participating in the IMOs, since we have also included some of their original solutions in this book.

The illustrations of geometry problems were done in WinGCLC, a program created by Prof. Predrag Janičić. This program is specifically designed for creating geometric pictures of unparalleled complexity quickly and efficiently. Even though it is still in its testing phase, its capabilities and utility are already remarkable and worthy of highest compliment.

Finally, we would like to thank our families for all their love and support during the making of this book.

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