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# EVALUATION AND DECISION MODELS WITH MULTIPLE CRITERIA

Stepping stones for the analyst

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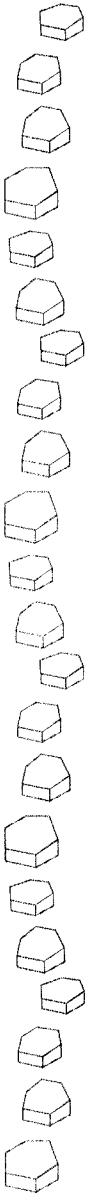
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# EVALUATION AND DECISION MODELS WITH MULTIPLE CRITERIA

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**Denis Bouyssou**  
*CNRS – LAMSADE*



**Thierry Marchant**  
*Universiteit Gent*



**Marc Pirlot**  
*Faculté Polytechnique de Mons*



**Alexis Tsoukiàs**  
*CNRS – LAMSADE*



**Philippe Vincke**  
*Université Libre de Bruxelles*

 **Springer**

Denis Bouyssou  
LAMSADE - CNRS  
Paris, France

Thierry Marchant  
Universiteit Gent  
Belgium

Marc Pirlot  
Polytechnique de Mons  
Belgium

Alexis Tsoukias  
LAMSADE - CNRS  
Paris, France

Philippe Vincke  
Universite Libre de Bruxelles  
Belgium

Library of Congress Control Number: 2005937803

ISBN-10: 0-387-31098-3 (HB) ISBN-10: 0-387-31099-1 (e-book)  
ISBN-13: 978-0387-31098-5 (HB) ISBN-13: 978-0387-31099-2 (e-book)

Printed on acid-free paper.

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Printed in the United States of America.

9 8 7 6 5 4 3 2 1

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