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John Stillwell

The Four Pillars of Geometry

With 138 Illustrations

Springer

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To Elaine

Preface

Many people think there is only one "right" way to teach geometry. For two millennia, the "right" way was Euclid's way, and it is still good in many respects. But in the 1950s the cry "Down with triangles!" was heard in France and new geometry books appeared, packed with linear algebra but with no diagrams. Was this the new "right" way, or was the "right" way something else again, perhaps transformation groups?

In this book, I wish to show that geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor. Euclid-style construction and axiomatics seem the best way to start, but linear algebra smooths the later stages by replacing some tortuous arguments by simple calculations. And how can one avoid projective geometry? It not only explains why objects look the way they do; it also explains why geometry is entangled with algebra. Finally, one needs to know that there is not one geometry, but many, and transformation groups are the best way to distinguish between them.

Two chapters are devoted to each approach: The first is concrete and introductory, whereas the second is more abstract. Thus, the first chapter on Euclid is about straightedge and compass constructions; the second is about axioms and theorems. The first chapter on linear algebra is about coordinates; the second is about vector spaces and the inner product. The first chapter on projective geometry is about perspective drawing; the second is about axioms for projective planes. The first chapter on transformation groups gives examples of transformations; the second constructs the hyperbolic plane from the transformations of the real projective line.

I believe that students are shortchanged if they miss any of these four approaches to the subject. Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing. (It is as if one discovered that number theory need not use addition and multiplication, but could be based on, say, the exponential function.)

The many faces of geometry are not only a source of amazement and delight. They are also a great help to the learner and teacher. We all know that some students prefer to visualize, whereas others prefer to reason or to calculate. Geometry has something for everybody, and all students will find themselves building on their strengths at some times, and working to overcome weaknesses at other times. We also know that Euclid has some beautiful proofs, whereas other theorems are more beautifully proved by algebra. In the multifaceted approach, every theorem can be given an elegant proof, and theorems with radically different proofs can be viewed from different sides.

This book is based on the course Foundations of Geometry that I taught at the University of San Francisco in the spring of 2004. It should be possible to cover it all in a one-semester course, but if time is short, some sections or chapters can be omitted according to the taste of the instructor. For example, one could omit Chapter 6 or Chapter 8. (But with regret, I am sure!)

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Thanks also go to my wife Elaine, who proofread the first version of the book, and to Robin Hartshorne, John Howe, Marc Ryser, Abe Shenitzer, and Michael Stillwell, who carefully read the revised version and saved me from many mathematical and stylistic errors.

Finally, I am grateful to the M. C. Escher Company – Baarn – Holland for permission to reproduce the Escher work *Circle Limit I* shown in Figure 8.19, and the explicit mathematical transformation of it shown in Figure 8.10. This work is copyright (2005) The M. C. Escher Company.

JOHN STILLWELL
San Francisco, November 2004
South Melbourne, April 2005

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