

Undergraduate Texts in Mathematics

Editors

S. Axler

K.A. Ribet

Undergraduate Texts in Mathematics

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(continued on page 228)

John Stillwell

The Four Pillars of Geometry

With 138 Illustrations

 Springer

John Stillwell
Department of Mathematics
University of San Francisco
San Francisco, CA 94117-1080
USA
stillwell@usfca.edu

Editorial Board

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA

K.A. Ribet
Department of Mathematics
University of California
at Berkeley
Berkeley, CA 94720-3840
USA

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To Elaine

Preface

Many people think there is only one “right” way to teach geometry. For two millennia, the “right” way was Euclid’s way, and it is still good in many respects. But in the 1950s the cry “Down with triangles!” was heard in France and new geometry books appeared, packed with linear algebra but with no diagrams. Was this the new “right” way, or was the “right” way something else again, perhaps transformation groups?

In this book, I wish to show that geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor. Euclid-style construction and axiomatics seem the best way to start, but linear algebra smooths the later stages by replacing some tortuous arguments by simple calculations. And how can one avoid projective geometry? It not only explains why objects look the way they do; it also explains why geometry is entangled with algebra. Finally, one needs to know that there is not one geometry, but many, and transformation groups are the best way to distinguish between them.

Two chapters are devoted to each approach: The first is concrete and introductory, whereas the second is more abstract. Thus, the first chapter on Euclid is about straightedge and compass constructions; the second is about axioms and theorems. The first chapter on linear algebra is about coordinates; the second is about vector spaces and the inner product. The first chapter on projective geometry is about perspective drawing; the second is about axioms for projective planes. The first chapter on transformation groups gives examples of transformations; the second constructs the hyperbolic plane from the transformations of the real projective line.

I believe that students are shortchanged if they miss any of these four approaches to the subject. Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer

to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing. (It is as if one discovered that number theory need not use addition and multiplication, but could be based on, say, the exponential function.)

The many faces of geometry are not only a source of amazement and delight. They are also a great help to the learner and teacher. We all know that some students prefer to visualize, whereas others prefer to reason or to calculate. Geometry has something for everybody, and all students will find themselves building on their strengths at some times, and working to overcome weaknesses at other times. We also know that Euclid has some beautiful proofs, whereas other theorems are more beautifully proved by algebra. In the multifaceted approach, every theorem can be given an elegant proof, and theorems with radically different proofs can be viewed from different sides.

This book is based on the course Foundations of Geometry that I taught at the University of San Francisco in the spring of 2004. It should be possible to cover it all in a one-semester course, but if time is short, some sections or chapters can be omitted according to the taste of the instructor. For example, one could omit Chapter 6 or Chapter 8. (But with regret, I am sure!)

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Finally, I am grateful to the M. C. Escher Company – Baarn – Holland for permission to reproduce the Escher work *Circle Limit I* shown in Figure 8.19, and the explicit mathematical transformation of it shown in Figure 8.10. This work is copyright (2005) The M. C. Escher Company.

JOHN STILLWELL
San Francisco, November 2004
South Melbourne, April 2005

Contents

Preface	vii
1 Straitgedge and compass	1
1.1 Euclid's construction axioms	2
1.2 Euclid's construction of the equilateral triangle	4
1.3 Some basic constructions	6
1.4 Multiplication and division	10
1.5 Similar triangles	13
1.6 Discussion	17
2 Euclid's approach to geometry	20
2.1 The parallel axiom	21
2.2 Congruence axioms	24
2.3 Area and equality	26
2.4 Area of parallelograms and triangles	29
2.5 The Pythagorean theorem	32
2.6 Proof of the Thales theorem	34
2.7 Angles in a circle	36
2.8 The Pythagorean theorem revisited	38
2.9 Discussion	42
3 Coordinates	46
3.1 The number line and the number plane	47
3.2 Lines and their equations	48
3.3 Distance	51
3.4 Intersections of lines and circles	53
3.5 Angle and slope	55
3.6 Isometries	57

3.7	The three reflections theorem	61
3.8	Discussion	63
4	Vectors and Euclidean spaces	65
4.1	Vectors	66
4.2	Direction and linear independence	69
4.3	Midpoints and centroids	71
4.4	The inner product	74
4.5	Inner product and cosine	77
4.6	The triangle inequality	80
4.7	Rotations, matrices, and complex numbers	83
4.8	Discussion	86
5	Perspective	88
5.1	Perspective drawing	89
5.2	Drawing with straightedge alone	92
5.3	Projective plane axioms and their models	94
5.4	Homogeneous coordinates	98
5.5	Projection	100
5.6	Linear fractional functions	104
5.7	The cross-ratio	108
5.8	What is special about the cross-ratio?	110
5.9	Discussion	113
6	Projective planes	117
6.1	Pappus and Desargues revisited	118
6.2	Coincidences	121
6.3	Variations on the Desargues theorem	125
6.4	Projective arithmetic	128
6.5	The field axioms	133
6.6	The associative laws	136
6.7	The distributive law	138
6.8	Discussion	140
7	Transformations	143
7.1	The group of isometries of the plane	144
7.2	Vector transformations	146
7.3	Transformations of the projective line	151
7.4	Spherical geometry	154

7.5	The rotation group of the sphere	157
7.6	Representing space rotations by quaternions	159
7.7	A finite group of space rotations	163
7.8	The groups \mathbb{S}^3 and \mathbb{RP}^3	167
7.9	Discussion	170
8	Non-Euclidean geometry	174
8.1	Extending the projective line to a plane	175
8.2	Complex conjugation	178
8.3	Reflections and Möbius transformations	182
8.4	Preserving non-Euclidean lines	184
8.5	Preserving angle	186
8.6	Non-Euclidean distance	191
8.7	Non-Euclidean translations and rotations	196
8.8	Three reflections or two involutions	199
8.9	Discussion	203
	References	213
	Index	215