S*pringer* **M***onographs in* **M***athematics*

Pekka Neittaanmaki Jürgen Sprekels
Dan Tiba

Optimization of Elliptic Systems

Theory and Applications

Springer

Pekka Neittaanmaki
Department of Mathematical
Information Technology
and
Department of Mathematics and Statistics
University of Jyvaskyla
Jyvaskyla, 40014
Finland
pn@mit.jyu.fi

Jürgen Sprekels
Angew. Analysis und Stochastik Abt. Part.
Differentialgleichg.
Weierstrass Institute for Applied Analysis
and Stochastics
Berlin, 10117
Germany
sprekels@wias-berlin.de

Dan Tiba
Institute of Mathematics
Romanian Academy
Bucuresti, 07070
Dan.Tiba@imar.ro

Mathematics Subject Classification (2000): 49-02, 49Q10, 35J85

ISBN-10: 0-387-27235-6 e-ISBN 0-387-27236-4
ISBN-13: 978-0387-27235-1

Printed on acid-free paper.

Printed in the United States of America. (SHER)

9 8 7 6 5 4 3 2 1

springeronline.com

Preface

The present monograph is intended to provide a comprehensive and accessible introduction to the optimization of elliptic systems. This area of mathematical research, which has many important applications in science and technology, has experienced an impressive development during the past two decades. There are already many good textbooks dealing with various aspects of optimal design problems. In this regard, we refer to the works of Pironneau [1984], Haslinger and Neittaanmäki [1988], [1996], Sokołowski and Zolésio [1992], Litvinov [2000], Allaire [2001], Mohammadi and Pironneau [2001], Delfour and Zolésio [2001], and Mäkinen and Haslinger [2003]. Already Lions [1968] devoted a major part of his classical monograph on the optimal control of partial differential equations to the optimization of elliptic systems. Let us also mention that even the very first known problem of the calculus of variations, the *brachistochrone* studied by Bernoulli back in 1696, is in fact a shape optimization problem.

The natural richness of this mathematical research subject, as well as the extremely large field of possible applications, has created the unusual situation that although many important results and methods have already been established, there are still pressing unsolved questions. In this monograph, we aim to address some of these open problems; as a consequence, there is only a minor overlap with the textbooks already existing in the field.

The exposition concentrates along two main directions:

- the optimal control of linear and nonlinear elliptic equations, including *variational inequalities* and *control into coefficients problems*,
- problems involving unknown and/or variable domains, like general *shape optimization problems* defined on various classes of bounded domains in Euclidean space, or *free boundary problems* arising in various physical processes.

It should be noted that many shape optimization problems occur naturally as control into coefficients problems. A large and interesting class of examples of this type, to which the whole of Chapter 6 is devoted, concerns the optimization of basic mechanical structures like beams, plates, arches, curved rods, and shells.

There are strong connections between all these seemingly different types of problems. This fact has for the first time been illustrated in the so-called *map-*

ping method introduced by Murat and Simon [1976], which makes it possible to transform domain optimization problems into control into coefficients problems. Throughout this monograph, we will try to elucidate such connections. Another classical contribution to the solution of shape optimization problems is the *speed method*, which was introduced by Zolésio [1979] and thoroughly discussed in the above-mentioned publications.

One basic feature of this textbook is the endeavor to relax the needed regularity assumptions as much as possible in order to include large classes of possible applications. We have succeeded in this aim for several fundamental questions:

- The existence theory for general domain optimization problems presented in Chapter 2 requires just the uniform continuity of the domain boundaries.
- The existence theory and the sensitivity analysis for plates and for curved mechanical structures, mainly performed in Chapter 6, is established under regularity hypotheses that are one or two degrees (depending on the case) lower than those usually postulated in the scientific literature.

Another characteristic of this book is that we have tried to stress the application of optimal control methods even in the case of problems involving variable/unknown domains. In this respect, it should be mentioned that our techniques are close to the works of Lions [1968], [1983], Cesari [1983], Barbu [1984], [1993], and Barbu and Precupanu [1986]. We are thoroughly convinced that optimal control theory may provide a rather complete and reliable approach to the challenging problems involving the optimization of systems defined on variable domains. Many of the presented results in this direction, mostly in Chapter 5, are original contributions of the authors.

In order to give the reader a comprehensive overview of the subject, we also report on other important results from the existing literature. Whenever certain theoretical developments are already available in textbook form, our discussion will be limited to the shortest possible presentation.

The book is organized in six chapters that give a gradual and accessible presentation of the material, where we have made a special effort to present numerous examples, both at the theoretical and at the numerical level. The material covers

- motivating examples of "purely" mathematical nature or originating from various applications (in Chapter 1),
- general existence results for control and shape optimization problems (in Chapter 2),
- a sensitivity analysis of linear and nonlinear control problems in the absence of differentiability assumptions, based on various penalization methods (in Chapter 3),

- the presentation of the a priori estimates technique for the numerical approximation of control problems governed by linear or nonlinear elliptic equations (in Chapter 4),
- optimal control and other approaches in unknown domain problems including free boundaries and optimal design (in Chapter 5),
- a fairly complete optimization theory of curved mechanical structures like arches, curved rods, and shells (in Chapter 6).

The three appendices collect important notions and results from the theory of function spaces and elliptic equations, from convex and nonlinear analysis, and from functional analysis, which are frequently used throughout this monograph.

In Chapters 5 and 6, several rather complex geometric optimization problems are studied in detail and are completely solved, including numerical results. We do not discuss the questions that arise from the practical implementation of the presented methods on a computer or from the solving of the associated finite-dimensional problems, as they do not enter into the objective of this book.

Let us also mention at this place that in order to keep the exposition at a reasonable length and due to other reasons, several directions of active research, such as second-order optimality conditions, a posteriori error estimates, homogenization methods, and applications of shape optimization in fluid mechanics, could not be covered in this textbook. However, we have tried to provide the reader with the corresponding relevant references in some of these subjects.

Now we comment briefly on some examples and applications, and we make a more detailed presentation of the text. The aim is to give the reader, from the very beginning, a clear image about the problems and the questions that are studied in this book, and about their motivation and difficulties.

We consider first the simplest case of an elastic shell of constant thickness that admits a general cylindrical surface as its midsurface. We assume that the shell is clamped along two of its generators and the forces acting on it are constant along the generators and perpendicular to them. Consequently, it is clear that the resulting deformation of the shell is also constant along the generators.

It is enough to investigate a two-dimensional section perpendicular to the generators. The obtained structure in $\mathbf{R}^2$ is called an arch, and its deformation is described by the so-called Kirchhoff–Love model. We mention bridges, roads, industrial tubes, windows, roofs, among others, as real-life examples entering this description. The design of such structures puts several important questions to the engineer or the architect: maximize the mechanical resistence of the structure, minimize the total cost, fulfill all the (technological) constraints that are imposed, etc. In general, a "compromise" among the sometimes conflicting aims has to be found.

We indicate now the mathematical formulation of the Kirchhoff–Love model. If $\varphi = (\varphi_1, \varphi_2) : [0, 1] \to \mathbf{R}^2$ is the parametrization of the arch with respect to its arc length and $c : [0,1] \to \mathbf{R}$ denotes its curvature, then the deformation vector $\bar{v} = (v_1, v_2) \in H_0^1(0,1) \times H_0^2(0,1)$ is the solution of

$$\int_0^1 \left[\frac{1}{\varepsilon}(v_1' - c\,v_2)(s)(u_1' - c\,u_2)(s) + (v_2' + c\,v_1)'(s)(u_2' + c_1\,u_1)'(s)\right] ds$$

$$= \int_0^1 (f_1\,u_1 + f_2\,u_2)(s)\,ds\,, \qquad \forall\, u_1 \in H_0^1(0,1)\,,\ \forall\, u_2 \in H_0^2(0,1)\,.$$

Here, $\sqrt{\varepsilon}$ represents the constant thickness of the arch and $[f_1, f_2] \in L^2(0,1)^2$ are, respectively, the tangential and normal components of the forces loading the clamped arch (assumed to act in its plane), while the tangential component v_1 and the normal component v_2 perform a similar representation for the deformation. The arbitrary functions $u_1 \in H_0^1(0,1)$ and $u_2 \in H_0^2(0,1)$ are test functions specific to the weak (variational) formulation of differential equations. Let us also mention that a complete study of this problem may be found in Ciarlet [1978, p. 432].

As the shape of the arch is completely characterized by its curvature c, the corresponding geometric optimization problems may be formulated as the minimization of some functional subject to the Kirchhoff–Love model as a side constraint and with the function c as the minimization parameter (control). For instance, one integral cost functional of interest is

$$\int_0^1 [v_2(s)]^2\,ds\,.$$

This means to find the form of the arch that has a minimal normal displacement in the sense of the above norm under the action of some known load (f_1, f_2). This is a natural safety requirement in many applications. Further (technological) constraints may be imposed directly on the admissible controls c or on the corresponding state (v_1, v_2).

We notice that the mere formulation of these problems requires the curvature c and its derivative (in the second term on the left side of the above equation). To ensure the integrability of such expressions one needs $\varphi \in W^{3,\infty}(0,1)^2$ or $\varphi \in C^3[0,1]^2$ for the corresponding parametrization. It is obvious that such requirements are inappropriate to the potential applications (see Figure 1.1 in Chapter 6, the Gothic arch). Moreover, some of the simplest and most popular discretization approaches (see Chapter 4) introduce nonsmooth approximations of φ in a natural way, and again the Kirchhoff–Love model cannot be applied. Such examples show that new mathematical methods have to be developed in order to relax the regularity hypotheses and to ensure a broad class of applications. In this book, a more sophisticated variational technique called the control variational method, based on control theory, is discussed. It is due to the authors and represents an alternative to the classical Dirichlet principle in the theory of elliptic equations. It is used for the analysis and optimization of Lipschitzian arches in Section 6.1 and of a simplified model of plates with discontinuous thickness in §3.4.2. More geometric optimization problems with mechanical background, such as optimal design of three-dimensional elastic curved rods and of general elastic shells, are studied by other methods in Sections 6.2 and 6.3. Thickness optimization problems for plates are investigated in §2.2.2 and Section 3.4. They are highly nonconvex optimization problems, but they still enjoy the property that they are defined in some known domain in the Euclidean space $\mathbf{R}^d$, $d \in \mathbf{N}$. In the above example, $d = 1$ and the domain is $]0,1[$.

We now present another example that involves unknown/variable domains. The application is related to the confinement of plasma in a tokamak machine. We denote by $\Omega \subset \mathbf{R}^2$ the smooth and bounded domain representing the cross section of the void chamber and by $D \subset \Omega$ its (unknown) subdomain occupied by the confined plasma (see Figure 2.1 in Chapter 1). Within the void region $\Omega \setminus \bar{D}$, the poloidal flux ψ satisfies (cf. Blum [1989, Ch. V]) the elliptic equation

$$-\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial \psi}{\partial y}\right) = 0 \quad \text{in } \Omega \setminus \bar{D},$$

which is nonsingular $(x > c > 0)$ due to the natural choice of coordinates, based on the symmetry of the tokamak in $\mathbf{R}^3$. The boundary ∂D of the plasma is one of the unknowns of the problem, and this is an example of a free boundary problem. In order to identify it, one uses supplementary measurements on the outer boundary $\partial\Omega$:

$$\psi = f\,, \quad \frac{1}{x}\frac{\partial \psi}{\partial n} = g \quad \text{on } \partial\Omega\,.$$

One can introduce a shape optimization problem with minimization parameter given by the unknown domain $D \subset \Omega$, with performance index

$$\int_{\partial\Omega} \left|\frac{1}{x}\frac{\partial \psi}{\partial n} - g\right|^2 d\tau$$

obtained by the penalization of the second boundary condition and with side conditions given by the first boundary condition and the elliptic equation for ψ in $\Omega \setminus \bar{D}$. This formulation can be further refined by introducing a fictitious control variable and a Tikhonov regularization as in Example 1.2.6 in Chapter 1. Other simple examples of variable domain optimization problems may be found in §2.3.1. In Section 5.1, the relationship between free boundary problems and shape optimization problems is further explored, while §5.3.1 presents the connection between variable domain problems and control into coefficients problems via the classical mapping and speed methods. Since such a procedure demands high regularity properties for the unknown domains, we introduce in Section 5.2 several alternative approaches, based on control theory, which may be applied in more general situations. Moreover, in Section 2.3 a rather complete existence theory for variable domains optimization problems is developed under the mere (uniform) continuity assumption for the unknown boundaries. In Sections 2.1, 2.2 (existence), and Chapter 3 (optimality conditions), a rather complete presentation of control problems for linear and nonlinear elliptic equations, including variational inequalities, is given.

Although all of us have been actively involved in the study of optimization problems in infinite-dimensional spaces for many years, the origin of this book can be traced back to the lectures delivered by one of us in 1995 during the summer school that is organized annually by the University of Jyväskylä. These lectures have been published in the form of the report Tiba [1995b]. The following ten years were marked by an intensive cooperation between us that is witnessed by the publication of numerous papers in all of the research directions forming the subject of this monograph.

Much of the material covered in this volume is original and resulted from our studies when we were affiliated with the University of Jyväskylä, the Humboldt University Berlin, the Institute for Mathematics of the Romanian Academy of Sciences in Bucharest, and the Weierstrass Institute in Berlin. The financial support of these institutions, of the Academy of Finland, of the Alexander-von-Humboldt Foundation, and of the DFG Research Center MATHEON in Berlin, is gratefully acknowledged.

This monograph is addressed to a large readership, primarily to master's or doctoral students and researchers working in this field of mathematics. Much of this material will prove useful also to scientists from other fields where the optimization of elliptic systems occurs, such as physics, mechanics, and engineering.

During the preparation of this monograph, we obtained much encouragement and many helpful hints from a number of colleagues who cannot be named here. We are also indebted to Springer-Verlag, especially to Achi Dosanjh (New York), for their continuing encouragement.

Finally, we would like to thank Marja-Leena Rantalainen (Jyväskylä) and Jutta Lohse (WIAS Berlin) for their efforts in the excellent LaTeX setting of this text. We are also indebted to Dipl.-Math. Gerd Reinhardt (WIAS Berlin) for his help in solving the problems arising from the inclusion of the figures in the text. Of course, the authors carry the full responsibility for each occasional misprint or other possible mistake in this monograph.

Jyväskylä, Berlin, and Bucharest, March 2005

P. Neittaanmäki, J. Sprekels, and D. Tiba

A Brief Reader's Guide

The authors are fully aware of the fact that the reader of this volume will usually be interested in only a certain part of it. Therefore, we give some hints in order to facilitate the reader's orientation within the text.

The book is divided into six chapters, referred to as Chapter 1 to Chapter 6, and three appendices, referred to as Appendix 1 to Appendix 3. Each of the chapters consists of several "sections," called Section 1.1, Section 6.1, and so on. The sections themselves may be divided into several subsections, called "paragraphs" and referred to, for example, as §3.1.3. Also, these paragraphs may have subparagraphs denoted, for instance, by §3.1.3.1. Clearly, the latter refers to the first subparagraph of the third paragraph in the first section of Chapter 3.

Let us also comment on the numbering used in this textbook. Equations are numbered by three integers that refer to the corresponding chapter, section, and equation, in that order. If, for example, we refer to equation (4.2.6), then we mean the sixth equation in the second section of Chapter 4. Definitions, Theorems, Lemmas, Propositions, Corollaries, and Examples, are also numbered sectionwise within each chapter; typical examples are Theorem 5.2.1, Lemma 6.2.4, Definition 2.2.1, and so on. An exception to this rule is the numbering within the three appendices, where references are made in the form Proposition A1.1, Theorem A2.3, Definition A3.1, and the like, with obvious meaning. Remarks are not numbered. Finally, figures are numbered sectionwise within each chapter.

Contents