
A POSTERIORI ERROR ANALYSIS VIA DUALITY THEORY

Advances in Mechanics and Mathematics

Volume 8

Series Editors:

David Y. Gao

Virginia Polytechnic Institute and State University, U.S.A.

Ray W. Ogden

University of Glasgow, U.K.

Advisory Editors:

I. Ekeland

University of British Columbia, Canada

K.R. Rajagopal

Texas A&M University, U.S.A.

T. Ratiu

Ecole Polytechnique, Switzerland

W. Yang

Tsinghua University, P.R. China

A POSTERIORI ERROR ANALYSIS VIA DUALITY THEORY

With Applications in Modeling and
Numerical Approximations

by

WEIMIN HAN
Department of Mathematics
University of Iowa
Iowa City, IA 52242, U.S.A.

 Springer

Library of Congress Cataloging-in-Publication Data

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 0-387-23536-1

e-ISBN 0-387-23537-X Printed on acid-free paper.

© 2005 Springer Science+Business Media, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, Inc., 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

SPIN 11336112

springeronline.com

Contents

List of Figures	vii
List of Tables	xi
Preface	xv
1. PRELIMINARIES	1
1.1 Introduction	1
1.2 Some basic notions from functional analysis	5
1.3 Function spaces	7
1.4 Weak formulation of boundary value problems	16
1.5 Best constants in some Sobolev inequalities	20
1.6 Singularities of elliptic problems on planar nonsmooth domains	25
1.7 An introduction of elliptic variational inequalities	29
1.8 Finite element method, error estimates	36
2. ELEMENTS OF CONVEX ANALYSIS, DUALITY THEORY	47
2.1 Convex sets and convex functions	47
2.2 Hahn–Banach theorem and separation of convex sets	50
2.3 Continuity and differentiability	52
2.4 Convex optimization	56
2.5 Conjugate functionals	57
2.6 Duality theory	59
2.7 Applications of duality theory in a posteriori error analysis	61
3. A POSTERIORI ERROR ANALYSIS FOR IDEALIZATIONS IN LINEAR PROBLEMS	67
3.1 Coefficient idealization	68
3.2 Right-hand side idealization	91

3.3	Boundary condition idealizations	100
3.4	Domain idealizations	106
3.5	Error estimates for material idealization of torsion problems	112
3.6	Simplifications in some heat conduction problems	119
4.	A POSTERIORI ERROR ANALYSIS FOR LINEARIZATIONS	127
4.1	Linearization of a nonlinear boundary value problem	127
4.2	Linearization of a nonlinear elasticity problem	143
4.3	Linearizations in heat conduction problems	160
4.4	Nonlinear problems with small parameters	169
4.5	A quasilinear problem	173
4.6	Laminar stationary flow of a Bingham fluid	176
4.7	Linearization in an obstacle problem	182
5.	A POSTERIORI ERROR ANALYSIS FOR SOME NUMERICAL PROCEDURES	193
5.1	A posteriori error analysis for regularization methods	193
5.2	Kačanov method for nonlinear problems	203
5.3	Kačanov method for a stationary conservation law	209
5.4	Kačanov method for a quasi-Newtonian flow problem	219
5.5	Application in solving an elastoplasticity problem	226
6.	ERROR ANALYSIS FOR VARIATIONAL INEQUALITIES OF THE SECOND KIND	235
6.1	Model problem and its finite element approximation	237
6.2	Dual formulation and a posteriori error estimation	243
6.3	Residual-based error estimates for the model problem	248
6.4	Gradient recovery-based error estimates for the model problem	255
6.5	Numerical example on the model problem	262
6.6	Application to a frictional contact problem	271
	REFERENCES	287
	Index	301

List of Figures

1.1	Numerical analysis of a physical problem	2
1.2	Smoothness of the boundary	10
1.3	Smooth domains	11
1.4	Lipschitz domains	11
1.5	A crack domain	11
1.6	A corner domain near the corner O	26
1.7	A finite element mesh	39
2.1	Continuity of a convex function	52
2.2	Subdifferential of the absolute value function	53
3.1	Choice of the auxiliary function near a corner domain: Subcase 2A	75
3.2	Choice of the auxiliary function near a corner domain: Subcase 2B	77
3.3	Domain idealization	109
4.1	One-dimensional stress-strain relation	131
4.2	Neighborhood around a corner	136
4.3	The function ϕ for numerical examples	153
4.4	Setting of a nonlinear elasticity problem	159
4.5	The working problem	159
5.1	Example 5.9, convergence of Kačanov iterates	219
5.2	Example 5.10, convergence of Kačanov iterates	220
6.1	Example 6.11, true solution	265
6.2	Example 6.11, initial partition and adaptively refined partition after 5 iterations	266
6.3	Example 6.11, $\ u - u_h^{un}\ _{1,\Omega}$ (\square) vs. $h^{1/2}\ \lambda - \lambda_h\ _{0,\Gamma}$ (\triangle)	267

6.4	Example 6.11, plots of $\lambda_{h,1}$	267
6.5	Example 6.11, plots of $\lambda_{h,2}$	268
6.6	Example 6.11, results based on residual type estimator, $\ u - u_h^{un}\ _{1,\Omega}$ (\square) vs. $\ u - u_h^{ad}\ _{1,\Omega}$ (\triangle)	269
6.7	Example 6.11, results based on gradient recovery type estimator, $\ u - u_h^{un}\ _{1,\Omega}$ (\square) vs. $\ u - u_h^{ad}\ _{1,\Omega}$ (\triangle)	269
6.8	Example 6.11, results based on residual type estimator, $\ u - u_h^{un}\ _{1,\Omega}$ and η_R on uniform mesh (\square) vs. $\ u - u_h^{ad}\ _{1,\Omega}$ and η_R on adapted mesh (\triangle)	270
6.9	Example 6.11, results based on gradient recovery type estimator, $\ u - u_h^{un}\ _{1,\Omega}$ and η_G on uniform mesh (\square) vs. $\ u - u_h^{ad}\ _{1,\Omega}$ and η_G on adapted mesh (\triangle)	270
6.10	Example 6.11, performance comparison between the two error estimators	271
6.11	Example 6.12, physical setting	274
6.12	Example 6.12, initial mesh with 128 elements, 81 nodes	275
6.13	Example 6.12, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $h^{1/2} \ \boldsymbol{\lambda}_{h\tau} - \boldsymbol{\lambda}_{h\tau}^{un}\ _{0,\Gamma_C}$ (\triangle)	276
6.14	Example 6.12, plot of $\lambda_{h,1}$	276
6.15	Example 6.12, plot of $\lambda_{h,2}$	277
6.16	Example 6.12, results based on residual type estimator, deformed configuration 5583 elements, 2921 nodes	277
6.17	Example 6.12, results based on gradient recovery type estimator, deformed configuration with 5437 elements, 2832 nodes	278
6.18	Example 6.12, results based on residual type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ (\triangle)	278
6.19	Example 6.12, results based on gradient recovery type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ (\triangle)	279
6.20	Example 6.12, results based on residual type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ and η_R on uniform mesh (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ and η_R on adapted mesh (\triangle)	279
6.21	Example 6.12, results based on gradient recovery type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ and η_G on uniform mesh (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ and η_G on adapted mesh (\triangle)	280
6.22	Example 6.13, physical setting	280
6.23	Example 6.13, initial mesh with 160 elements, 105 nodes	281
6.24	Example 6.13, results based on residual type estimator, deformed configuration with 5222 elements, 2755 nodes	281

6.25	Example 6.13, results based on gradient recovery type estimator, deformed configuration with 4964 elements, 2624 nodes	281
6.26	Example 6.13, plot of $\lambda_{h,1}$	282
6.27	Example 6.13, plot of $\lambda_{h,2}$	282
6.28	Example 6.13, results based on residual type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ (\triangle)	283
6.29	Example 6.13, results based on gradient recovery type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ (\triangle)	283
6.30	Example 6.13, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ (\square) vs. $h^{1/2}\ \boldsymbol{\lambda}_{h\tau} - \boldsymbol{\lambda}_{h\tau}^{un}\ _{0,\Gamma_C}$ (\triangle)	284
6.31	Example 6.13, results based on residual type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ and η_R on uniform mesh (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ and η_R on adapted mesh (\triangle)	284
6.32	Example 6.13, results based on gradient recovery type estimator, $\ \mathbf{u} - \mathbf{u}_h^{un}\ _{\mathbf{V}}$ and η_G on uniform mesh (\square) vs. $\ \mathbf{u} - \mathbf{u}_h^{ad}\ _{\mathbf{V}}$ and η_G on adapted mesh (\triangle)	285

List of Tables

3.1	Example 3.8, effectivity of error bound	89
3.2	Example 3.10, effectivity of error bounds	91
3.3	Example 3.12, smallest roots for several angles	98
4.1	Example 4.1, true errors and error bounds, $\lambda = 0.1$	140
4.2	Example 4.1, true errors and error bounds, $\lambda = 0.015$	140
4.3	Example 4.1, true errors and error bounds, $\lambda = 0.0101$	140
4.4	Example 4.2, error bounds for problems on an L-shape domain	142
4.5	Singular exponents	157
4.6	Example 4.11, error bounds for various parameters	160
4.7	Example 4.12, error bound for energy difference	177
4.8	Efficiency of error bounds on a one-dimensional obstacle problem	192
5.1	Example 5.7, numerical results with $\alpha(s) = (2+s)/(1+s)$, $\beta_0 = 7/8$	215
5.2	Example 5.7, numerical results with $\alpha(s) = (7+s)/(1+s)$, $\beta_0 = 1/4$	215
5.3	Example 5.7, numerical results with $\alpha(s) = (2+s^2)/(1+s^2)$, $\beta_0 = 7/16$	216
5.4	Example 5.8, numerical results with $\kappa = 0.20488$	217
5.5	Example 5.8, numerical results with $\kappa = 10^{-1}$	217
5.6	Example 5.8, numerical results with $\kappa = 10^{-2}$	217
5.7	Example 5.8, numerical results with $\kappa = 10^{-3}$	217
5.8	Example 5.8, numerical results with $\kappa = 10^{-4}$	218
5.9	Example 5.9, convergence of Kačanov iterates	218
5.10	Example 5.10, convergence of Kačanov iterates	220

5.11	Example 5.11, Kačanov iteration for a quasi-Newtonian flow problem, $\eta_0 = 1, \eta_\infty = 0.0001$	226
5.12	Example 5.11, Kačanov iteration for a quasi-Newtonian flow problem, $\eta_0 = 100, \eta_\infty = 0.01$	226
6.1	Nodes and weights of a 7-point Gauss-Legendre quadrature formula over the reference triangle	263
6.2	Example 6.11, numerical values of constants	268

Dedicated to

DAQING HAN, SUZHEN QIN

HUIDI TANG

ELIZABETH, MICHAEL

Preface

This work provides a posteriori error analysis for mathematical idealizations in modeling boundary value problems, especially those arising in mechanical applications, and for numerical approximations of numerous nonlinear variational problems. An error estimate is called a posteriori if the computed solution is used in assessing its accuracy. A posteriori error estimation is central to measuring, controlling and minimizing errors in modeling and numerical approximations. In this book, the main mathematical tool for the developments of a posteriori error estimates is the duality theory of convex analysis, documented in the well-known book by Ekeland and Temam ([49]). The duality theory has been found useful in mathematical programming, mechanics, numerical analysis, etc.

The book is divided into six chapters. The first chapter reviews some basic notions and results from functional analysis, boundary value problems, elliptic variational inequalities, and finite element approximations. The most relevant part of the duality theory and convex analysis is briefly reviewed in Chapter 2. This brief review is sufficient for the applications of the duality theory in all the following chapters. In mathematical modeling of differential equation problems, usually assumptions are made on various data. Qualitatively, for many problems, it is known that the solution depends continuously on the problem data. Frequently though, it is desirable also to estimate or bound quantitatively the effect on the solutions of the problems caused by the adoption of the assumptions on the data. In Chapter 3, a posteriori error estimates are derived for the effect on the solutions of mathematical idealizations on the data of elliptic linear boundary value problems. In Chapter 4, a posteriori error estimates are given for linearization in a number of nonlinear boundary value problems. The last two chapters are devoted to a posteriori error analysis of numerical solutions. In Chapter 5, the regularization method and the Kačanov method are considered, both being useful in handling certain types of nonlinearity. In Chapter 6, a posteriori error estimates are derived and studied for finite element solutions of some elliptic variational inequalities.

This book is intended for researchers and graduate students in Applied and Computational Mathematics, and Engineering. Mathematical prerequisites include calculus, linear algebra, some exposures of differential equations, and concepts of normed spaces, Banach spaces and Hilbert spaces. In the theoretical development, some basic notions and results in functional analysis, duality theory, weak formulations of boundary value problems, variational inequalities, and the finite element method are used. Brief reviews of these notions and

results in the first two chapters provide background materials for a reader who lacks knowledge in these areas.

This work avoids giving the results in the most general, abstract form so that it is easier for the reader to understand more clearly the essential ideas involved. Many examples are included to show the usefulness of the derived error estimates.

In preparing this book, I have benefited from many individuals. I am grateful to Professor Ivo Babuška for introducing me the research topic and for providing valuable advice. Several of my collaborators (teachers, friends, and students) made contributions to various parts of the book, and I especially thank Dr. Viorel Bostan, Dr. Jiuhua Chen, Professor Hongci Huang, late Professor Søren Jensen, Professor B.D. Reddy. I express my gratitude to Professor Kendall Atkinson and Professor Mircea Sofonea for their constant support. I thank Professor D.Y. Gao and Professor R.W. Ogden for inviting me to make the contribution in their Kluwer book series on Advances in Mechanics and Mathematics (AMMA).

The supports of NSF under grant DMS-0106781 and the James Van Allen Fellowship of the University of Iowa are greatly appreciated.