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*(continued after index)*

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# Weyl Transforms

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# Preface

This book is an outgrowth of courses given by me for graduate students at York University in the past ten years. The actual writing of the book in this form was carried out at York University, Peking University, the Academia Sinica in Beijing, the University of California at Irvine, Osaka University, and the University of Delaware. The idea of writing this book was first conceived in the summer of 1989, and the protracted period of gestation was due to my daily duties as a professor at York University. I would like to thank Professor K.C. Chang, of Peking University; Professor Shujie Li, of the Academia Sinica in Beijing; Professor Martin Schechter, of the University of California at Irvine; Professor Michihiro Nagase, of Osaka University; and Professor M.Z. Nashed, of the University of Delaware, for providing me with stimulating environments for the exchange of ideas and the actual writing of the book.

We study in this book the properties of pseudo-differential operators arising in quantum mechanics, first envisaged in [33] by Hermann Weyl, as bounded linear operators on $L^2(\mathbb{R}^n)$. Thus, it is natural to call the operators treated in this book Weyl transforms.

To be specific, my original plan was to supplement the standard graduate course in pseudo-differential operators at York University by writing a set of lecture notes on the derivation of a formula from first principles for the product of two Weyl transforms. This was achieved in the summer of 1990 when I was visiting Peking University and the Academia Sinica in Beijing. Chapters 2–6 of the book, which appeared then, albeit in embryonic form, already contained the formula for the product of two Weyl transforms obtained by Pool in [20]. Chapters 8 and 9 were written in the summer of 1993 at York University in order to get another formula for the product of two Weyl transforms using relatively new ideas, e.g., the Heisenberg group and the twisted convolution, in noncommutative harmonic analysis developed by Folland in [6] and Stein in [26], among others. The result was an account, given in Chapter 9, of a formula for the product of two Weyl transforms in the paper [10] by Grossmann, Loupias, and Stein. A preliminary version of the derivations of the two formulas was written up for private circulation in the second quarter of 1994–95 at the University of California at Irvine.

In the summer of 1994, I gave a course in special topics in pseudo-differential operators tailored to the needs of my Ph.D. students at York University. I chose to study the criteria in terms of the symbols for the boundedness and compactness of the Weyl transforms. Two sets of results were presented. The first set was about the compactness of a Weyl transform with symbol in $L^r(\mathbb{R}^{2n})$, $1 \le r \le \infty$, and the second set, inspired by the book [29] by Thangavelu, was concerned with the criteria for the boundedness and compactness of Weyl transforms in terms of symbols evaluated at Wigner transforms of Hermite functions. The two sets of results can be found in, respectively, Chapters 11–14 and Chapters 24–27. Chapter 28 is devoted to the study of the eigenvalues and eigenfunctions of a Weyl transform of which the symbol is a Dirac delta on a disk in $\mathbb{R}^2$.

The preliminary version of the formulas for the product of two Weyl transforms and the lecture notes of the topics course given in the summer of 1994 were then put together, simplified, polished, and supplemented with background materials at Osaka University and the University of Delaware in the winter of 1997. To this end, I found it instructive to add new chapters, i.e., Chapters 15–17, on localization operators initiated by Daubechies in [3, 4] and Daubechies and Paul in [5], and the closely related theory of square-integrable group representations studied by Grossmann, Morlet, and Paul in [11, 12]. The final two chapters were added in an attempt to make explicit the role of the symplectic group in the study of Weyl transforms.

The connections of the Weyl transforms with quantization in physics, highlighted in this book, can be found in the references [6, 10, 20, 26, 33] already cited, the book [2] by Berezin and Shubin, the paper [18] by Iancu and Wong, and the papers [37, 38] by Wong.

All the topics in this book should be accessible to a first-year graduate student. The book is a natural sequel to a first course in pseudo-differential operators, but no familiarity with even the basics of pseudo-differential operators is required for a good understanding of the entire book. The only essential prerequisites are the elementary properties of the Fourier transform and tempered distributions given in the beginning chapters of, say, the book [8] by Goldberg, the book [27] by Stein and Weiss, and the book [36] by Wong, and these are collected in Chapter 1. Of course, a nodding acquaintance with basic functional analysis is necessary for an intelligent reading of this book.

Finally, it must be emphasized that this book is far from being a definitive treatise on Weyl transforms. Thus, the choice of topics in this book was guided by personal predilections, and the references at the end of the book are limited to those that have been instrumental in my understanding of Weyl transforms.

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