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Preface

This book goes back a long way. There is a tradition of research and teaching in inelasticity at Stanford that goes back at least to Wilhelm Flügge and Erastus Lee. I joined the faculty in 1980, and shortly thereafter the Chairman of the Applied Mechanics Division, George Herrmann, asked me to present a course in plasticity. I decided to develop a new two-quarter sequence entitled “Theoretical and Computational Plasticity” which combined the basic theory I had learned as a graduate student at the University of California at Berkeley from David Bogy, James Kelly, Jacob Lubliner, and Paul Naghdi with new computational techniques from the finite-element literature and my personal research. I taught the course a couple of times and developed a set of notes that I passed on to Juan Simo when he joined the faculty in 1985. I was Chairman at that time and I asked Juan to further develop the course into a full year covering inelasticity from a more comprehensive perspective. Juan embarked on this path creating what was to become his signature course. He eventually renamed it “Computational and Theoretical Inelasticity” and it covered much of the material that was the basis of his research in material modeling and simulation for which he achieved international recognition. At the outset we decided to write a book that would cover the material in the course. The first draft was written quite expeditiously, and versions of it have been circulated privately among friends, colleagues, and interested members of the research community since 1986. Thereafter progress was intermittent and slow. Some things were changed and some new chapters were added, but we both had become distracted by other activities in the early 1990s. Prior to that, we frequently discussed what would be necessary “to get it out the door,” but I do not recall the subject even coming up once in the years immediately preceding Juan’s death in 1994. Since that time I have been repeatedly urged to bring the project to completion. Through the efforts of a number of individuals, the task is now completed.

This book describes the theoretical foundations of inelasticity, its numerical formulation, and implementation. It is felt that the subject matter described herein constitutes a representative sample of state-of-the-art methodology currently used in inelastic calculations. On the other hand, no attempt has been made to present a careful account of the historical developments of the subject or to examine in detail important physical aspects underlying inelastic flow in solids. Likewise, the

list of references should, by no means, be regarded as a complete literature survey of the field.

Chapter 1 begins with an overview of small deformation plasticity and viscoplasticity in a one-dimensional setting. Notions introduced in Chapter 1 are generalized to multiple dimensions and developed more comprehensively in subsequent chapters. Ideas of convex optimization theory, which are the foundations of the numerical implementation of plasticity, are first introduced in Chapter 1. In Chapter 2 the theory is generalized to multiple dimensions. In addition to the three-dimensional case, plane-strain and plane-stress cases are presented, as well as thermodynamic considerations and the principle of maximal plastic dissipation. Chapter 3 deals with integration algorithms for the constitutive equations of plasticity and viscoplasticity. The two most important classes of return-mapping algorithms are described, namely, the closest-point projection and cutting-plane algorithms. The classical radial return method is also presented. Another important mathematical tool in the construction of numerical methods for inelastic constitutive equations, the operator-splitting methodology, is also introduced in Chapter 3. Chapter 4 deals with the variational setting of boundary-value problems and discretization by finite element methods. Key technologies for successful implementation of inelasticity, such as the assumed strain method and the B-bar approach, are described. The generalization of the theory to nonsmooth yield surfaces is considered in Chapter 5. Mathematical numerical analysis issues of general return-mapping algorithms and, in particular, their nonlinear stability are presented in Chapter 6. The generalization to finite-strain inelasticity theory commences in Chapter 7 with an introduction to nonlinear continuum mechanics, the notion of objectivity, variational formulations of the large-deformation case, and hyperelastic and hypoelastic constitutive equations. The practically important subject of objective integrative algorithms for rate constitutive equations is described in Chapter 8. In Chapter 9 the theory of hyperelastic-based plasticity models is presented. This chapter covers the local multiplicative decomposition of the deformation gradient into elastic and plastic parts and numerical formulations of this concept by way of return-mapping algorithms. Chapter 10 deals with small and large deformation viscoelasticity.

I believe a good, basic course of a semester's or quarter's duration would focus on Chapter 1 to 4. For more advanced students wishing to understand the large deformation theory, Chapters 7 and 8 are essential. Chapter 8, in particular, deals with the types of formulations commonly used in large-scale commercial computer programs. There is more research interest in the hyperelastic-based theories of Chapter 9, which are more satisfying from a theoretical point of view. However, as of this writing, they have not enjoyed similar attention from the developers of most commercial computer programs.

Over the past two years, this text has been used as the basis of courses at Stanford and Berkeley which provided vehicles for readying the manuscript for publication. I wish to sincerely thank the students in these classes for their considerable patience and effort. Present and past graduate students of Juan's and mine were also instrumental in bringing the endeavor to fruition. Among them I wish to thank, in

particular, Francisco Armero, Krishnakumar Garikipati, Sanjay Govindjee, John Kennedy, and Steve Rifai. However, without the hard work and devotion of two recent students, I doubt that this project would have been completed: Vinay Rao and Eva Petőcz critically read the manuscript and interacted with the other individuals who provided corrections. Vinay and Eva synthesized the inputs, made changes, and managed the master file containing the manuscript. They searched for and found lost drawings, and when missing figures could not be located, they drew them themselves. They spent many hours in this effort, and I wish to express my sincere thanks and gratitude to them.

Thomas J. R. Hughes
Stanford, March 1998

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