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Asymptotic Cones and Functions in Optimization and Variational Inequalities

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This book is dedicated to

Martine, François, and Jérôme
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Preface

Nonlinear applied analysis and in particular the related fields of continuous optimization and variational inequality problems have gone through major developments over the last three decades and have reached maturity. A pivotal role in these developments has been played by convex analysis, a rich area covering a broad range of problems in mathematical sciences and its applications. Separation of convex sets and the Legendre–Fenchel conjugate transforms are fundamental notions that have laid the ground for these fruitful developments. Two other fundamental notions that have contributed to making convex analysis a powerful analytical tool and that have often been hidden in these developments are the notions of asymptotic sets and functions.

The purpose of this book is to provide a systematic and comprehensive account of asymptotic sets and functions, from which a broad and useful theory emerges in the areas of optimization and variational inequalities. There is a variety of motivations that led mathematicians to study questions revolving around attaintment of the infimum in a minimization problem and its stability, duality and minmax theorems, convexification of sets and functions, and maximal monotone maps. In all these topics we are faced with the central problem of handling unbounded situations. This is particularly true when standard compactness hypotheses are not present. The appropriate concepts and tools needed to study such kinds of problems are vital not only in theory but also within the development of numerical methods. For the latter, we need not only to prove that a sequence generated by a given algorithm is well defined, namely an existence result, but also to establish that the produced sequence remains bounded. One can seldom directly apply theorems of classical analysis to answer to such questions. The notions of asymptotic cones and associated asymptotic functions provide a natural and unifying framework to resolve these types of problems. These notions have been used mostly and traditionally in convex analysis, with many results scattered in the literature. Yet these concepts also have a prominent and independent role to play in both convex and nonconvex analysis. This book presents the material reflecting this last point with many parts, including new results and covering convex and nonconvex problems. In particular, our aim is to demonstrate not only the interplay between classical convex-analytic results and the asymptotic machinery, but also the wide potential of the latter in analyzing variational problems.

We expect that this book will be useful to graduate students at an advanced level as well as to researchers and practitioners in the fields of optimization theory, nonlinear programming, and applied mathematical sciences. We decided to use a style with detailed and often transparent proofs. This might sometimes bore the more advanced reader, but should at least make the reading of the book easier and hopefully even enjoyable. The material is presented within the finite-dimensional setting. Our motivation for this choice was to eliminate the obvious complications that would have emerged within a more general topological setting and would have obscured the stream of the main ideas and results. For the more advanced reader, it is noteworthy to realize that most of the notions and properties developed here can be easily extended to reflexive Banach Spaces, assuming a supplementary condition with respect to weak convergence. The extension to more general arbitrary topological spaces is certainly not obvious, but the finite-dimensional setting is rich enough to motivate the interested reader toward the development of corresponding results needed in areas such as partial differential equations and probability analysis.

Structure of the Book

In Chapter 1 we recall the basic mathematical background: elementary convex analysis and set-valued maps. The results are presented without proofs. This material is classical and can be skipped by anyone who has had a standard course in convex analysis. None of this chapter's results rely on any asymptotic notions. Chapter 2 is the heart of the book and gives the fundamental results on asymptotic cones and functions. The interplay between geometry and analysis is emphasized and will be followed consistently in the remaining chapters. Building on the concept of asymptotic cone of the epigraph of a function, the notion of asymptotic function emerges, and calculus at infinity can be developed. The role of asymptotic functions in formulating general optimization problems is described. Chapter 3 studies the existence of optimal solutions for general optimization problems and related stability results, and also demonstrates the power of the asymptotic results developed in Chapter 2. Standard results under coercivity and weak coercivity assumptions imply that the solution set is a nonempty compact set and the sum of a compact set with a linear space, respectively. Here we develop many new properties for the noncoercive and weakly coercive cases through the use of asymptotic sets to derive more general existence results with applications leading to some new theorems "à la Helly" and for the convex feasibility problems. In Chapter 4 we study the subject of minimizing stationary sequences and error bounds. Both topics are central in the study of numerical methods. The concept of well-behaved asymptotic functions and the properties of such functions, which in turn is linked to the problems of error bounds associated with a given subset of a Euclidean space, are introduced. A general framework is developed around these two themes to characterize asymptotic optimality and error bounds for convex inequality systems. Duality theory plays a fundamental role in optimization and is developed in Chapter 5. The abstract perturbational scheme, valid for any optimization problem, is the starting point of the analysis. Under a minimal set of assumptions and thanks to asymptotic calculus, we derive key duality results, which are then applied to cover the classical Lagrange and Fenchel duality as well as minimax theorems, in a simple and unified way. Chapter 6 provides a self-contained introduction to maximal monotone maps and variational inequalities. Solving a convex optimization problem is reduced to solving a generalized equation associated with the subdifferential map. In many areas of applied mathematics, game theory, and equilibrium problems in economy, generalized equations arise and are described in terms of more general maps, in particular maximal monotone maps. The chapter covers the classical material together with some more recent results, streamlining the role of asymptotic functions.

Each chapter ends with some bibliographical notes and references. We did not attempt to give a complete bibliography on the covered topics, which is rather large, and we apologize in advance for any omission in the cited references. Yet, we have tried to cite all the sources that have been used in this book as well as some significant original historical developments, together with more recent references in the field that should help to guide researchers for further reading.

The book can be used as a complementary text to graduate courses in applied analysis and optimization theory. It can also serve as a text for a topics course at the graduate level, based, for example, on Chapters 2, 3, and 5, or as an introduction to variational inequality problems through Chapter 6, which is essentially self-contained.

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