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Serge Lang

Introduction to Differentiable Manifolds

Second Edition

With 12 Illustrations

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Foreword

This book is an outgrowth of my *Introduction to Differentiable Manifolds* (1962) and *Differential Manifolds* (1972). Both I and my publishers felt it worth while to keep available a brief introduction to differential manifolds.

The book gives an introduction to the basic concepts which are used in differential topology, differential geometry, and differential equations. In differential topology, one studies for instance homotopy classes of maps and the possibility of finding suitable differentiable maps in them (immersions, embeddings, isomorphisms, etc.). One may also use differentiable structures on topological manifolds to determine the topological structure of the manifold (for example, à la Smale [Sm 67]). In differential geometry, one puts an additional structure on the differentiable manifold (a vector field, a spray, a 2-form, a Riemannian metric, ad lib.) and studies properties connected especially with these objects. Formally, one may say that one studies properties invariant under the group of differentiable automorphisms which preserve the additional structure. In differential equations, one studies vector fields and their integral curves, singular points, stable and unstable manifolds, etc. A certain number of concepts are essential for all three, and are so basic and elementary that it is worthwhile to collect them together so that more advanced expositions can be given without having to start from the very beginnings. The concepts are concerned with the general basic theory of differential manifolds. My *Fundamentals of Differential Geometry* (1999) can then be viewed as a continuation of the present book.

Charts and local coordinates. A chart on a manifold is classically a representation of an open set of the manifold in some euclidean space. Using a chart does not necessarily imply using coordinates. Charts will be used systematically.

I don't propose, of course, to do away with local coordinates. They are useful for computations, and are also especially useful when integrating differential forms, because the $dx_1 \wedge \cdots \wedge dx_n$. corresponds to the $dx_1 \cdots dx_n$ of Lebesgue measure, in oriented charts. Thus we often give the local coordinate formulation for such applications. Much of the literature is still covered by local coordinates, and I therefore hope that the neophyte will thus be helped in getting acquainted with the literature. I also hope to convince the expert that nothing is lost, and much is gained, by expressing one's geometric thoughts without hiding them under an irrelevant formalism.

Since this book is intended as a text to follow advanced calculus, say at the first year graduate level or advanced undergraduate level, manifolds are assumed finite dimensional. Since my book *Fundamentals of Differential Geometry* now exists, and covers the infinite dimensional case as well, readers at a more advanced level can verify for themselves that there is no essential additional cost in this larger context. I am, however, following here my own admonition in the introduction of that book, to assume from the start that all manifolds are finite dimensional. Both presentations need to be available, for mathematical and pedagogical reasons.

New Haven 2002 Serge Lang

Acknowledgments

I have greatly profited from several sources in writing this book. These sources are from the 1960s.

First, I originally profited from Dieudonné's *Foundations of Modern Analysis*, which started to emphasize the Banach point of view.

Second, I originally profited from Bourbaki's *Fascicule de résultats* [Bou 69] for the foundations of differentiable manifolds. This provides a good guide as to what should be included. I have not followed it entirely, as I have omitted some topics and added others, but on the whole, I found it quite useful. I have put the emphasis on the differentiable point of view, as distinguished from the analytic. However, to offset this a little, I included two analytic applications of Stokes' formula, the Cauchy theorem in several variables, and the residue theorem.

Third, Milnor's notes [Mi 58], [Mi 59], [Mi 61] proved invaluable. They were of course directed toward differential topology, but of necessity had to cover ad hoc the foundations of differentiable manifolds (or, at least, part of them). In particular, I have used his treatment of the operations on vector bundles (Chapter III, §4) and his elegant exposition of the uniqueness of tubular neighborhoods (Chapter IV, §6, and Chapter VII, §4).

Fourth, I am very much indebted to Palais for collaborating on Chapter IV, and giving me his exposition of sprays (Chapter IV, §3). As he showed me, these can be used to construct tubular neighborhoods. Palais also showed me how one can recover sprays and geodesics on a Riemannian manifold by making direct use of the canonical 2-form and the metric (Chapter VII, §7). This is a considerable improvement on past expositions.

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