

Combinatorial Designs

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Combinatorial Designs

Constructions and Analysis



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To Ron Mullin, who taught me design theory

Foreword

The evolution of combinatorial design theory has been one of remarkable successes, unanticipated applications, deep connections with fundamental mathematics, and the desire to produce order from apparent chaos. While some of its celebrated successes date from the eighteenth and nineteenth centuries in the research of Euler, Kirkman, Cayley, Hamilton, Sylvester, Moore, and others, not until the twentieth century did the study of combinatorial designs emerge as an academic subject in its own right. When Fisher and his colleagues developed the mathematics of experimental design in the 1920s, combinatorial design theory was born as a field intimately linked to its applications. Beginning in the 1930s, Bose and his school laid the foundations, embedding the nascent field firmly as a mathematical discipline by developing deep connections with finite geometry, number theory, finite fields, and group theory; however, Bose accomplished much more. His foundation entwined deep mathematics with its applications in experimental design and in recreational problems and anticipated its fundamental importance in the theory of error-correcting codes.

The rapid advances in design theory can be attributed in large degree to its impetus from applications in coding theory and communications and its continued deep interactions with geometry, algebra, and number theory. The last fifty years have witnessed not only the emergence of certain combinatorial designs (balanced incomplete block designs, Hadamard matrices, pairwise balanced designs, and orthogonal arrays, for example) as central, but also powerful combinatorial and computational techniques for their construction. Indeed the field grew so far and so fast that its historical connection to applications was strained.

Yet, in the last twenty years, combinatorial design theory has emerged again as a field rich in current and practical applications. The fundamental connections with algebra, number theory, and finite geometry remain and flourish. The applications in experimental design and coding theory have developed a breadth and depth that defy brief explanation. Yet combinatorial design theory has matured into more than this through applications in

cryptography, optical communications, storage system design, communication protocols, algorithm design and analysis, and wireless communications, to mention just a few areas.

Combinatorial design theory is mature and widely applied today because it has respected and advanced its mathematical heritage while finding genuine new applications. I am honored to write this foreword for two reasons. Doug Stinson has for twenty-five years been the epitome of a researcher and expositor who has advanced combinatorial design theory as a marriage of mathematics and applications. But more than that, the book you hold in your hands presents design theory as a seamless interaction of deep mathematics and challenging applications. By providing an accessible introduction, it serves as an invitation to those in applications areas to appreciate and employ beautiful mathematics and concurrently invites mathematicians to learn from the applications themselves.

In which directions will combinatorial design theory evolve in the next century? We cannot yet know. We can know, however, that new mathematical truths will be found and that unanticipated applications will arise. Our challenge is to seek both and to know that each profits from the other.

Phoenix, Arizona
April, 2003

Charles J. Colbourn

Preface

Overview and Goals

Combinatorial design theory is one of the most beautiful areas of mathematics. Design theory has its roots in recreational mathematics, but it evolved in the twentieth century into a full-fledged mathematical discipline with diverse applications in statistics and computer science. The fundamental problems in design theory are simple enough that they can be explained to non-mathematicians, yet the solutions of those problems have involved the development of innovative new combinatorial techniques as well as ingenious applications of methods from other areas of mathematics such as algebra and number theory. Many classical problems remain unsolved to this day as well.

This book is intended primarily to be a textbook for study at the senior undergraduate or beginning graduate level. Courses in mathematics or computer science can be based on this book. Regardless of the audience, however, it requires a certain amount of “mathematical maturity” to study design theory. The main technical prerequisites are some familiarity with basic abstract algebra (group theory, in particular), linear algebra (matrices and vector spaces), and some number-theoretic fundamentals (e.g., modular arithmetic and congruences).

Topic Coverage and Organization

The first seven chapters of this book provide a thorough treatment of the classical core of the subject of combinatorial designs. These chapters concern symmetric BIBDs, difference sets, Hadamard matrices, resolvable BIBDs, Latin squares, and pairwise balanced designs. A one-semester course can cover most of this material. For example, when I have taught courses on designs, I have based my lectures on material selected from the following chapters and sections:

- Chapter 1: Sections 1.1–1.3, Section 1.4 (optional), Sections 1.5–1.6
- Chapter 2: Sections 2.1–2.4
- Chapter 3: Sections 3.1–3.4
- Chapter 4: Sections 4.1–4.4, Section 4.5 (optional), Section 4.6
- Chapter 5: Sections 5.1–5.2, Section 5.3 (optional)
- Chapter 6: Sections 6.1, Section 6.2 (optional), Sections 6.3–6.8
- Chapter 7: Sections 7.1–7.3

There are many variations possible, of course. Typically, I would provide a complete proof of the Bruck-Ryser-Chowla Theorem or the Multiplier Theorem, but not both. It is possible to omit Wilson's Construction for MOLS in order to spend more time on pairwise balanced designs. Another option is to include the optional Section 6.2 and omit some of the material in Chapter 7. Yet another possibility is to present an introduction to t -designs (incorporating some material from Chapter 9, Sections 9.1 and 9.2) and delete some of the optional sections listed above.

More advanced or specialized material is covered in the last four chapters as well as in some later sections of the first seven chapters. The main topics in the last four chapters are minimal pairwise balanced designs, t -designs, orthogonal arrays and codes, and four selected applications of designs (in the last chapter).

Key Features

There are several features of this book that will make it useful as a textbook. Complete, carefully written proofs of most major results are given. There are many examples provided throughout in order to illustrate the definitions, concepts, and theorems. Numerous and varied exercises are provided at the end of each chapter. As well, certain mathematical threads flow through this book:

- The linear algebraic method of proving Fisher's Inequality reappears several times.
- The theme of Boolean functions is introduced in the study of bent functions and revisited in the discussion of Reed-Muller codes and a brief treatment of resilient functions.
- The use of permutation groups as a construction technique is pervasive.
- Elegant combinatorial arguments are used in many places in preference to alternative proofs that employ heavier mathematical machinery.
- Finite fields are used throughout the book. For this reason, some background material on finite fields is summarized in an Appendix. However, another option for an instructor is to specialize constructions utilizing finite fields \mathbb{F}_q to the more familiar fields \mathbb{Z}_p , where p is a prime.

As mentioned earlier, there are a variety of advanced or specialized topics that are discussed in the book. Highlights include the following:

- regular Hadamard matrices and excess of Hadamard matrices;
- bent functions;
- bounds and constructions for minimal pairwise balanced designs;
- the Ryser-Woodall Theorem;
- constructions and bounds for t -wise balanced designs, including a proof of the Kramer Conjecture;
- a survey of the combinatorial connections between orthogonal arrays, codes, and designs;
- constructions and bounds for various classes of optimal codes and orthogonal arrays;
- Reed-Muller codes;
- resilient functions;
- four selected applications of designs: authentication codes, threshold schemes, group testing, and two-point sampling.

It must be recognized that design theory is an enormous subject, and any choice of optional material in a 300 page book is dependent on the whim of the author! Thus there are many interesting or important areas of design theory that are not discussed in the book. I hope, however, that readers of the book will find a fascinating mix of topics that serve to illustrate the breadth and beauty of design theory.

Audience

As mentioned above, this book is primarily intended to be a textbook. In addition, all of the material in this book is suitable for self-study by graduate students, who will find it provides helpful background information concerning research topics in design theory. Researchers may also find that some of the sections on advanced topics provide a useful reference for material that is not easily accessible in textbook form.

Acknowledgments

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Contents

Foreword	VII
Preface	IX
1 Introduction to Balanced Incomplete Block Designs	1
1.1 What Is Design Theory?	1
1.2 Basic Definitions and Properties	2
1.3 Incidence Matrices	6
1.4 Isomorphisms and Automorphisms	8
1.4.1 Constructing BIBDs with Specified Automorphisms ...	12
1.5 New BIBDs from Old	15
1.6 Fisher's Inequality	16
1.7 Notes and References	18
1.8 Exercises	19
2 Symmetric BIBDs	23
2.1 An Intersection Property	23
2.2 Residual and Derived BIBDs	25
2.3 Projective Planes and Geometries	27
2.4 The Bruck-Ryser-Chowla Theorem	30
2.5 Notes and References	39
2.6 Exercises	39
3 Difference Sets and Automorphisms	41
3.1 Difference Sets and Automorphisms	41
3.2 Quadratic Residue Difference Sets	50
3.3 Singer Difference Sets	52
3.4 The Multiplier Theorem	54
3.4.1 Multipliers of Difference Sets	54
3.4.2 The Group Ring	58
3.4.3 Proof of the Multiplier Theorem	61

3.5	Difference Families	63
3.6	A Construction for Difference Families	66
3.7	Notes and References	69
3.8	Exercises	70
4	Hadamard Matrices and Designs	73
4.1	Hadamard Matrices	73
4.2	An Equivalence Between Hadamard Matrices and BIBDs	74
4.3	Conference Matrices and Hadamard Matrices	76
4.4	A Product Construction	80
4.5	Williamson's Method	81
4.6	Existence Results for Hadamard Matrices of Small Orders	84
4.7	Regular Hadamard Matrices	84
4.7.1	Excess of Hadamard Matrices	87
4.8	Bent Functions	89
4.9	Notes and References	98
4.10	Exercises	98
5	Resolvable BIBDs	101
5.1	Introduction	101
5.2	Affine Planes and Geometries	102
5.2.1	Resolvability of Affine Planes	104
5.2.2	Projective and Affine Planes	106
5.2.3	Affine Geometries	107
5.3	Bose's Inequality and Affine Resolvable BIBDs	109
5.3.1	Symmetric BIBDs from Affine Resolvable BIBDs	114
5.4	Orthogonal Resolutions	115
5.5	Notes and References	119
5.6	Exercises	120
6	Latin Squares	123
6.1	Latin Squares and Quasigroups	123
6.2	Steiner Triple Systems	126
6.2.1	The Bose Construction	127
6.2.2	The Skolem Construction	128
6.3	Orthogonal Latin Squares	131
6.4	Mutually Orthogonal Latin Squares	136
6.4.1	MOLS and Affine Planes	136
6.4.2	MacNeish's Theorem	139
6.5	Orthogonal Arrays	140
6.5.1	Orthogonal Arrays and MOLS	140
6.5.2	Some Constructions for Orthogonal Arrays	142
6.6	Transversal Designs	144
6.7	Wilson's Construction	146
6.8	Disproof of the Euler Conjecture	151

6.9	Notes and References	153
6.10	Exercises	153
7	Pairwise Balanced Designs I	157
7.1	Definitions and Basic Results	157
7.2	Necessary Conditions and PBD-Closure	159
7.3	Steiner Triple Systems	164
7.4	$(v, 4, 1)$ -BIBDs	167
7.5	Kirkman Triple Systems	170
7.6	Notes and References	176
7.7	Exercises	177
8	Pairwise Balanced Designs II	179
8.1	The Stanton-Kalbfleisch Bound	179
8.1.1	The Erdős-de Bruijn Theorem	183
8.2	Improved Bounds	185
8.2.1	Some Examples	188
8.3	Minimal PBDs and Projective Planes	190
8.4	Minimal PBDs with $\lambda > 1$	193
8.5	Notes and References	198
8.6	Exercises	198
9	t-Designs and t-wise Balanced Designs	201
9.1	Basic Definitions and Properties of t -Designs	201
9.2	Some Constructions for t -Designs with $t \geq 3$	206
9.2.1	Inversive Planes	209
9.2.2	Some 5-Designs	212
9.3	t -wise Balanced Designs	216
9.3.1	Holes and Subdesigns	217
9.4	Notes and References	221
9.5	Exercises	222
10	Orthogonal Arrays and Codes	225
10.1	Orthogonal Arrays	225
10.2	Codes	230
10.3	Bounds on Codes and Orthogonal Arrays	233
10.4	New Codes from Old	236
10.5	Binary Codes	239
10.5.1	The Plotkin Bound and Hadamard Codes	239
10.5.2	Reed-Muller Codes	242
10.6	Resilient Functions	249
10.7	Notes and References	253
10.8	Exercises	253

11 Applications of Combinatorial Designs	257
11.1 Authentication Codes	257
11.1.1 A Construction from Orthogonal Arrays	259
11.2 Threshold Schemes	261
11.2.1 A Construction from Orthogonal Arrays	261
11.2.2 Anonymous Threshold Schemes	263
11.3 Group Testing Algorithms	264
11.3.1 A Construction from BIBDs	266
11.4 Two-Point Sampling	268
11.4.1 Monte Carlo Algorithms	268
11.4.2 Orthogonal Arrays and Two-Point Sampling	270
11.5 Notes and References	273
11.6 Exercises	273
A Small Symmetric BIBDs and Abelian Difference Sets	279
B Finite Fields	281
References	287
Index	295