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# Hankel Operators and Their Applications



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# Preface

The purpose of this book is to describe the theory of Hankel operators, one of the most important classes of operators on spaces of analytic functions. Hankel operators can be defined as operators having infinite Hankel matrices (i.e., matrices with entries depending only on the sum of the coordinates) with respect to some orthonormal basis. Finite matrices with this property were introduced by Hankel, who found interesting algebraic properties of their determinants. One of the first results on infinite Hankel matrices was obtained by Kronecker, who characterized Hankel matrices of finite rank as those whose entries are Taylor coefficients of rational functions. Since then Hankel operators (or matrices) have found numerous applications in classical problems of analysis, such as moment problems, orthogonal polynomials, etc.

Hankel operators admit various useful realizations, such as operators on spaces of analytic functions, integral operators on function spaces on  $(0, \infty)$ , operators on sequence spaces. In 1957 Nehari described the bounded Hankel operators on the sequence space  $\ell^2$ . This description turned out to be very important and started the contemporary period of the study of Hankel operators.

We begin the book with introductory Chapter 1, which defines Hankel operators and presents their basic properties. We consider different realizations of Hankel operators and important connections of Hankel operators with the spaces  $BMO$  and  $VMO$ , Sz.-Nagy–Foias functional model, reproducing kernels of the Hardy class  $H^2$ , moment problems, and Carleson imbedding operators.

It turns out that for the needs of applications it is also important to consider vectorial Hankel operators, i.e., Hankel operators on spaces of vector functions. We introduce vectorial Hankel operators in Chapter 2, to be used later in the book in control theory, approximation theory, and Wiener–Hopf factorizations (Chapters 11, 13, and 14).

In Chapter 3 we introduce another very important class of operators on spaces of analytic functions, the class of Toeplitz operators. They can be defined as operators having infinite matrices with entries depending only on the difference of the coordinates. Though Hankel and Toeplitz operators have quite different properties, Hankel operators play an important role in the study of Toeplitz operators, and vice versa. We also study in Chapter 3 vectorial Toeplitz operators.

In Chapter 4 we analyze the singular values of Hankel operators. The main result of the chapter is the Adamyan–Arov–Krein theorem, which shows that the  $n$ th singular value of a Hankel operator is the distance to the set of Hankel operators of rank at most  $n$ .

Chapter 5 deals with parametrization of solutions of the Nehari problem. In other words, we parametrize the symbols of a Hankel (or a vectorial Hankel) operator that belong to the ball in  $L^\infty$  of a given radius.

In Chapter 6 we describe the Hankel operators that belong to Schatten–von Neumann classes  $S_p$  as those whose symbols belong to certain Besov classes. We consider various applications of this description. In particular we obtain sharp results on rational approximation in the norm of *BMO*.

In this book we study many different applications of Hankel operators (in approximation theory, prediction theory, interpolation problems, control theory, etc). In Chapters 8 and 9 we use Hankel operators to study regularity conditions for stationary processes.

Chapter 10 is an introduction to the spectral theory of Hankel operators. We continue the analysis of the spectral problems of Hankel operators in Chapter 12, where we give a complete description of the spectral properties of the self-adjoint Hankel operators. It turns out that not only are Hankel operators used in control theory, but also the theory of Hankel operators can benefit from methods of control theory. In particular, in Chapter 12 the results on spectral properties of self-adjoint Hankel operators are based on balanced linear systems with continuous time and discrete time, a notion borrowed from control theory.

Chapter 11 is devoted to applications of Hankel operators in control theory. We consider linear systems with discrete time and continuous time, the problems of robust stabilization, model reduction, and model matching.

In Chapter 13 we study hereditary properties of maximizing vectors of vectorial Hankel operators. In other words, for a broad class of function spaces  $X$  we prove that if the symbol belongs to  $X$ , then all maximizing vectors belong to the same space  $X$ . We give several applications of this result. In particular, we use it to obtain hereditary properties of Wiener–Hopf factorizations.

In Chapter 14 vectorial Hankel operators are used in the theory of approximation by analytic matrix and operator functions. We introduce the important notion of superoptimal approximation and prove the uniqueness of a superoptimal approximation under certain mild conditions on the matrix function. We obtain certain special factorizations and prove inequalities between the singular values of the corresponding Hankel operators and superoptimal singular values of their symbols. This beautiful theory has been developed during the last decade; it demonstrates the importance of vectorial Hankel operators in noncommutative analysis.

One of the most beautiful applications of Hankel operators is given in Chapter 15. The last chapter gives a solution to the famous problem of whether a polynomially bounded operator on Hilbert space must be similar to a contraction. This problem remained open for a long time. In particular, it was one of the problems in a famous paper by Paul Halmos called “Ten problems in Hilbert space”. Recently it has been solved in the negative with the help of vectorial Hankel operators.

In this book we discuss only classical Hankel operators (i.e., operators with Hankel matrices or, in other words, Hankel operators on the Hardy class  $H^2$ ). For the last 20 years many interesting results have been obtained about various generalizations of Hankel operators (commutators of multiplications and Calderón–Zygmund operators, paracommutators, Hankel operators on Bergman spaces, Hankel operators on function spaces on the polydisk, on the unit ball in  $\mathbb{C}^n$ , on classical domains, etc). However, it is physically impossible to cover such generalizations in one book, and we restrict ourselves to classical Hankel operators.

Even under this constraint it is hardly possible to cover all aspects of Hankel operators and their applications (for example, this book does not include applications of Hankel operators in noncommutative geometry, perturbation theory, or asymptotics of Toeplitz determinants). Each chapter ends with Concluding Remarks, where the reader can find references to some results not included here.

Theorems, lemmas, and corollaries (as well as displayed formulas) are numbered lexicographically. Within the same chapter Theorem 3.5 means the fifth item of Section 3. To refer to a result from a different chapter, we use three numbers: Lemma 5.5.4 means the fourth item of Section 5 in Chapter 5. Reference to §6 means Section 6 within the same chapter. Reference to §4.3 means Section 3 in Chapter 4. Displayed formulas have an independent numeration.

For convenience we add two appendices in which the reader can find necessary information on operator theory and function spaces. Reference to Appendix 2.5 means Section 5 of Appendix 2.

I would like to express my deep gratitude to my colleagues with whom I discussed many aspects of Hankel operators and their applications. I am especially grateful to A.B. Aleksandrov, R.B. Alexeev, E.M. Dyn’kin,

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# Notation

Symbols  $\mathbb{Z}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  denote the set of integers, real numbers, and complex numbers.

$\mathbb{R}_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x \geq 0\}$ ,  $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{n \in \mathbb{Z} : n \geq 0\}$ .

$\mathbb{T} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ ,  $\mathbb{D} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ .

$z$  denotes the identity map of  $\mathbb{C}$  (or a subset of  $\mathbb{C}$ ) onto itself.

$\mathbf{m}$  is normalized Lebesgue measure on  $\mathbb{T}$ .

$\mathbf{m}_2$  is planar Lebesgue measure.

$\hat{\varphi}(n)$  is the  $n$ th Fourier coefficients of  $\varphi$ .

$L^p$  means the  $L^p$  space of functions on  $\mathbb{T}$  with respect to  $\mathbf{m}$ , unless otherwise specified.

$\mathcal{B}(X, Y)$  for Banach spaces  $X$  and  $Y$  is the space of bounded linear operators from  $X$  to  $Y$ ,  $\mathcal{B}(X) \stackrel{\text{def}}{=} \mathcal{B}(X, X)$ .

$H^p$  is the Hardy class of functions analytic in  $\mathbb{D}$  (see Appendix 2).

If  $X$  is a Banach space,  $L^p(X)$  and  $H^p(X)$  denote the  $X$ -valued  $L^p$  and  $H^p$  spaces.

If  $E$  is a subset of  $\mathbb{R}$  or  $\mathbb{T}$ ,  $L^p(E)$  denotes the  $L^p$ -space of functions on  $E$ ; this should not lead to a confusion with the previous notation.

$H^p(\mathbb{C}_+)$  ( $H^p(\mathbb{C}_-)$ ,  $H^p(\mathbb{C}^+)$ , or  $H^p(\mathbb{C}_-)$ ) denotes the Hardy class of functions analytic in the upper (lower, right, or left) half-plane.

If  $E$  is a subset of  $\mathbb{R}$  or  $\mathbb{T}$  and  $X$  is a Banach space, we use the notation  $L^p(E, X)$  for the  $L^p$ -space of  $X$ -valued functions on  $E$ .

$\mathcal{F}$  is the Fourier transform,  $(\mathcal{F}f)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(s)e^{-2\pi i s t} ds$ ,  $f \in L^1(\mathbb{R})$ .

span  $A$  means the closed linear span of a subset  $E$  in a Banach space.

$\mathbf{1}$  is the constant function identically equal to 1.

$\mathbf{0}$  is the zero in a vector space.

$\mathcal{P}$  is the space of trigonometric polynomials of the form  $\sum c_j z^j$ .

$\mathcal{P}_+$  is the space of analytic polynomials of the form  $\sum_{j \geq 0} c_j z^j$ .

$\mathcal{P}_-$  is the space of antianalytic polynomials of the form  $\sum_{j < 0} c_j z^j$ .

If  $X$  is a space of functions on  $\mathbb{T}$  such that  $X \subset L^1$  (or  $X$  is a space of distributions on  $\mathbb{T}$ ), we put

$$X_+ \stackrel{\text{def}}{=} \{f \in X : \hat{f}(j) = 0 \text{ for } j < 0\}$$

and

$$X_- \stackrel{\text{def}}{=} \{f \in X : \hat{f}(j) = 0 \text{ for } j \geq 0\}.$$

$\mathcal{S}_p$  is the Schatten–von Neumann class (see Appendix 1).

$\mathcal{C} = \mathcal{S}_\infty$  is the space of compact operators on Hilbert space.

$\ell^2_{\mathbb{Z}}$  is the space of two-sided sequences  $\{x_n\}_{n \in \mathbb{Z}}$  such that  $\sum_{n \in \mathbb{Z}} |x_n|^2 < \infty$ .

All Hilbert spaces are assumed separable.

Unless otherwise stated, an operator means a bounded linear operator.

# Contents

|  |     |
|--|-----|
| <b>Chapter 1. An Introduction to Hankel Operators</b> .....    | 1   |
| 1. Bounded Hankel Operators .....                              | 2   |
| 2. Hankel Operators and Compressed Shift .....                 | 13  |
| 3. Hankel Operators of Finite Rank .....                       | 19  |
| 4. Interpolation Problems .....                                | 23  |
| 5. Compactness of Hankel Operators .....                       | 25  |
| 6. Hankel Operators and Reproducing Kernels .....              | 37  |
| 7. Hankel Operators and Moment Sequences .....                 | 39  |
| 8. Hankel Operators as Integral Operators on the Semi-Axis ... | 46  |
| Concluding Remarks .....                                       | 56  |
| <b>Chapter 2. Vectorial Hankel Operators</b> .....             | 61  |
| 1. Completing Matrix Contractions .....                        | 62  |
| 2. Bounded Block Hankel Matrices .....                         | 66  |
| 3. Hankel Operators and the Commutant Lifting Theorem .....    | 71  |
| 4. Compact Vectorial Hankel Operators .....                    | 74  |
| 5. Vectorial Hankel Operators of Finite Rank .....             | 76  |
| 6. Imbedding Theorems .....                                    | 81  |
| Concluding Remarks .....                                       | 84  |
| <b>Chapter 3. Toeplitz Operators</b> .....                     | 87  |
| 1. Basic Properties .....                                      | 88  |
| 2. A General Invertibility Criterion .....                     | 94  |
| 3. Spectra of Certain Toeplitz Operators .....                 | 97  |
| 4. Toeplitz Operators on Spaces of Vector Functions .....      | 103 |

|  |            |
|--|------------|
| 5. Wiener–Hopf Factorizations of Symbols of Fredholm Toeplitz Operators .....              | 109        |
| 6. Left Invertibility of Bounded Analytic Matrix Functions ....                            | 119        |
| Concluding Remarks .....   | 121        |
| <b>Chapter 4. Singular Values of Hankel Operators .....</b>                                | <b>125</b> |
| 1. The Adamyan–Arov–Krein Theorem .....  | 126        |
| 2. The Case $s_m(\Gamma) = s_\infty(\Gamma)$ .....   | 131        |
| 3. Finite Rank Approximation of Vectorial Hankel Operators ..                              | 135        |
| 4. Relations between $H_u$ and $H_{\bar{u}}$ .....   | 142        |
| Concluding Remarks .....   | 145        |
| <b>Chapter 5. Parametrization of Solutions of the Nehari Problem ..</b>                    | <b>147</b> |
| 1. Adamyan–Arov–Krein Parametrization in the Scalar Case ..                                | 148        |
| 2. Parametrization of Solutions of the Nevanlinna–Pick Problem .....                       | 170        |
| 3. Parametrization of Solutions of the Nehari–Takagi Problem ..                            | 173        |
| 4. Parametrization via One-Step Extension .....  | 187        |
| 5. Parametrization in the General Case .....   | 212        |
| Concluding Remarks .....   | 228        |
| <b>Chapter 6. Hankel Operators and Schatten–von Neumann Classes</b>                        | <b>231</b> |
| 1. Nuclearity of Hankel Operators .....  | 232        |
| 2. Hankel Operators of Class $S_p$ , $1 < p < \infty$ .....                                | 239        |
| 3. Hankel Operators of Class $S_p$ , $0 < p < 1$ .....                                     | 243        |
| 4. Hankel Operators and Schatten–Lorentz Classes .....                                     | 253        |
| 5. Projecting onto the Hankel Matrices .....   | 257        |
| 6. Rational Approximation .....  | 267        |
| 7. Other Applications of the $S_p$ Criterion .....   | 275        |
| 8. Generalized Hankel Matrices .....   | 281        |
| 9. Generalized Block Hankel Matrices and Vectorial Hankel Operators .....                  | 292        |
| Concluding Remarks .....   | 296        |
| <b>Chapter 7. Best Approximation by Analytic and Meromorphic Functions .....</b>           | <b>303</b> |
| 1. Function Spaces That Can Be Described in Terms of Rational Approximation in $BMO$ ..... | 306        |
| 2. Best Approximation in Decent Banach Algebras .....                                      | 312        |
| 3. Best Approximation in Spaces without a Norm .....                                       | 316        |
| 4. Examples and Counterexamples .....  | 318        |
| 5. Badly Approximable Functions .....  | 338        |
| 6. Perturbations of Multiple Singular Values of Hankel Operators .....                     | 341        |
| 7. The Boundedness Problem .....   | 344        |

|   |            |
|---|------------|
| 8. Arguments of Unimodular Functions .....  | 355        |
| 9. Schmidt Functions of Hankel Operators .....  | 356        |
| 10. Continuity in the sup-Norm .....  | 359        |
| 11. Continuity in Decent Banach Spaces .....  | 362        |
| 12. The Recovery Problem in Spaces of Measures and Interpolation<br>by Analytic Functions ..... | 371        |
| 13. The Fefferman–Stein Decomposition in $B_p^{1/p}$ .....                                      | 376        |
| Concluding Remarks .....  | 376        |
| <b>Chapter 8. An Introduction to Gaussian Spaces .....</b>                                      | <b>379</b> |
| 1. Gaussian Spaces .....  | 380        |
| 2. The Fock Space .....   | 384        |
| 3. Mixing Properties and Regularity Conditions .....  | 390        |
| 4. Minimality and Basisness .....   | 402        |
| 5. Scattering Systems and Hankel Operators .....  | 405        |
| 6. Geometry of Past and Future .....  | 408        |
| Concluding Remarks .....  | 412        |
| <b>Chapter 9. Regularity Conditions for Stationary Processes .....</b>                          | <b>415</b> |
| 1. Minimality in Spectral Terms .....   | 415        |
| 2. Angles between Past and Future .....   | 417        |
| 3. Regularity Conditions in Spectral Terms .....  | 420        |
| 4. Stronger Regularity Conditions .....   | 424        |
| Concluding Remarks .....  | 426        |
| <b>Chapter 10. Spectral Properties of Hankel Operators .....</b>                                | <b>431</b> |
| 1. The Essential Spectrum of Hankel Operators with Piecewise<br>Continuous Symbols .....        | 433        |
| 2. The Carleman Operator .....  | 440        |
| 3. Quasinilpotent Hankel Operators .....  | 443        |
| Concluding Remarks .....  | 450        |
| <b>Chapter 11. Hankel Operators in Control Theory .....</b>                                     | <b>453</b> |
| 1. Transfer Functions .....   | 454        |
| 2. Realizations with Discrete Time .....  | 457        |
| 3. Realizations with Continuous Time .....  | 466        |
| 4. Model Reduction .....  | 472        |
| 5. Robust Stabilization .....   | 473        |
| 6. Coprime Factorization .....  | 476        |
| 7. Proof of Theorem 5.1 .....   | 479        |
| 8. Parametrization of Stabilizing Controllers .....   | 482        |
| 9. Solution of the Robust Stabilization Problem. The Model<br>Matching Problem .....            | 485        |
| Concluding Remarks .....  | 487        |

|  |     |
|--|-----|
| <b>Chapter 12. The Inverse Spectral Problem for Self-Adjoint Hankel Operators</b> .....                  | 489 |
| 1. Necessary Conditions .....  | 493 |
| 2. Eigenvalues of Hankel Operators .....   | 497 |
| 3. Linear Systems with Continuous Time and Lyapunov Equations .....                                      | 499 |
| 4. Construction of a Linear System with Continuous Time .....  | 503 |
| 5. The Kernel of $\Gamma_h$ .....  | 509 |
| 6. Proofs of Lemmas 4.1 and 5.2 .....  | 516 |
| 7. Positive Hankel Operators with Multiple Spectrum .....  | 518 |
| 8. Moduli of Hankel Operators, Past and Future, and the Inverse Problem for Rational Approximation ..... | 520 |
| 9. Linear Systems with Discrete Time .....   | 522 |
| 10. Passing to Balanced Linear Systems .....   | 526 |
| 11. Asymptotic Stability .....   | 529 |
| 12. The Main Construction .....  | 531 |
| 13. Proof of Theorem 9.1 .....   | 536 |
| 14. Proofs of Lemmas 13.2 and 13.5 .....   | 542 |
| 15. A Theorem in Perturbation Theory .....   | 548 |
| Concluding Remarks .....   | 550 |
| <br>   |     |
| <b>Chapter 13. Wiener–Hopf Factorizations and the Recovery Problem</b> .....                             | 553 |
| 1. The Recovery Problem in $\mathcal{R}$ -spaces .....   | 554 |
| 2. Maximizing Vectors of Vectorial Hankel Operators .....  | 555 |
| 3. Wiener–Masani Factorizations .....  | 557 |
| 4. Isometric-Outer Factorizations .....  | 560 |
| 5. The Recovery Problem and Wiener–Hopf Factorizations of Unitary-Valued Functions .....                 | 561 |
| 6. Wiener–Hopf Factorizations. The General Case .....  | 562 |
| Concluding Remarks .....   | 563 |
| <br>   |     |
| <b>Chapter 14. Analytic Approximation of Matrix Functions</b> .....                                      | 565 |
| 1. Balanced Matrix Functions .....   | 568 |
| 2. Parametrization of Best Approximations .....  | 574 |
| 3. Superoptimal Approximation of $H^\infty + C$ Matrix Functions .....                                   | 578 |
| 4. Superoptimal Approximation of Matrix Functions $\Phi$ with Small Essential Norm of $H_\Phi$ .....     | 580 |
| 5. Thematic Factorizations and Very Badly Approximable Functions .....                                   | 586 |
| 6. Admissible and Superoptimal Weights .....   | 592 |
| 7. Thematic Indices .....  | 594 |
| 8. Inequalities Involving Hankel and Superoptimal Singular Values .....                                  | 600 |
| 9. Invariance of Residual Entries .....  | 602 |

|   |            |
|---|------------|
| 10. Monotone Thematic Factorizations and Invariance of Thematic Indices ..... | 607        |
| 11. Construction of Superoptimal Approximation and the Corona Problem .....   | 615        |
| 12. Hereditary Properties of Superoptimal Approximation .....                 | 618        |
| 13. Continuity Properties of Superoptimal Approximation .....                 | 623        |
| 14. Unitary Interpolants of Matrix Functions .....                            | 636        |
| 15. Canonical Factorizations .....  | 643        |
| 16. Very Badly Approximable Unitary-Valued Functions .....                    | 661        |
| 17. Superoptimal Meromorphic Approximation .....                              | 662        |
| 18. Analytic Approximation of Infinite Matrix Functions .....                 | 672        |
| 19. Back to the Adamyan–Arov–Krein Parametrization .....                      | 681        |
| Concluding Remarks .....  | 682        |
| <b>Chapter 15. Hankel Operators and Similarity to a Contraction ..</b>        | <b>685</b> |
| 1. Operators $R_\psi$ in the Scalar Case .....                                | 687        |
| 2. Power Bounded Operators $R_\psi$ .....                                     | 693        |
| 3. Counterexamples .....  | 694        |
| Concluding Remarks .....  | 703        |
| Appendix 1 .....  | 705        |
| Appendix 2 .....  | 717        |
| References .....  | 739        |
| Author Index .....  | 776        |
| Subject Index .....   | 780        |